

Localization and centrality in networks

Travis Martin,¹ Xiao Zhang,² and M. E. J. Newman^{2,3}

¹*Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109*

²*Department of Physics, University of Michigan, Ann Arbor, MI 48109*

³*Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48109*

Eigenvector centrality is a widely used measure of the importance of nodes in a network. Here we show that under common conditions the eigenvector centrality displays a localization transition that causes most of the weight of the centrality to concentrate on a small number of nodes in the network and renders the measure useless for most practical purposes. As a remedy, we propose an alternative centrality measure based on the non-backtracking matrix, which gives results closely similar to the standard eigenvector centrality in dense networks where the latter is well behaved, but avoids localization and gives useful results in regimes where the standard centrality fails.

In the study of networked systems such as social, biological, and technological networks, centrality is one of the most fundamental of measures. Centrality quantifies how important or influential a node is within a network. There are many different measures of centrality, the simplest being *degree centrality*, or simply degree, which is the number of connections a node has to other nodes. In a social network of acquaintances, for example, someone who knows many people is likely to be more influential than someone who knows few or none. *Eigenvector centrality* is a more sophisticated variant of the same idea, which recognizes that not all acquaintances are equal [1]. If the people you know are themselves influential, it makes you more influential too. Eigenvector centrality defines a centrality score v_i for each node i in an undirected network which is proportional to the sum of the scores of the node's network neighbors $v_i = \lambda^{-1} \sum_j A_{ij} v_j$, where λ is a constant and the sum is over all nodes. Here A_{ij} is an element of the adjacency matrix \mathbf{A} of the network having value one if there is an edge between nodes i and j and zero otherwise. Defining a vector \mathbf{v} whose elements are the v_i , we then have $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, meaning that the vector of centralities is an eigenvector of the adjacency matrix. If we further stipulate that the centralities should all be nonnegative, then it follows by the Perron–Frobenius theorem that \mathbf{v} must be the leading eigenvector (the vector corresponding to the most positive eigenvalue λ). Eigenvector centrality and its many variants are some of the most widely used of all centrality measures. They are commonly used in social network analysis [2] and form the basis for ranking algorithms such as Google's PageRank [3], the HITS algorithm [4], and the eigenfactor metric [5].

As we argue in this paper, however, eigenvector centrality also has serious flaws. In particular, we show that, depending on the details of the network structure, the leading eigenvector of the adjacency matrix can undergo a *localization transition*, akin to localization in disordered condensed-matter systems, in which most of the weight of the vector concentrates around one or a few nodes in the network, significantly diminishing the effectiveness of the vector as a tool for quantifying centrality. Moreover, we

show that localization can happen under common real-world conditions, for instance in networks with power-law degree distributions.

As a solution to these problems, we propose a new centrality measure based on the leading eigenvector of the so-called Hashimoto or non-backtracking matrix [6, 7]. This measure has the desirable properties of (1) being roughly equal to the standard eigenvector centrality in dense networks, where the latter is well behaved, while also (2) avoiding localization, and hence giving useful results, in regimes where the standard centrality fails.

Signs of localization in networks have been noted previously in numerical calculations of the eigenvectors of real-world networks [8, 9]. Here we demonstrate the existence and properties of the localization transition with analytic calculations based on techniques drawn from random matrix theory. Consider the following simple model network, which is a special case of a model introduced in [10] consisting of a random graph plus a single hub node. In a network of n nodes, $n - 1$ of them form a random graph in which every distinct pair of nodes is connected by an edge with independent probability $c/(n - 2)$, where c is the mean degree. The n th node is the hub and is connected to every other node with independent probability $d/(n - 1)$, so that the expected degree of the hub is d . In the regime where $c \gg 1$ [11] it is known that (with high probability) the spectrum of the random graph alone has the classic Wigner semicircle form, centered around zero, plus a single leading eigenvalue with value $c + 1$ and corresponding leading eigenvector equal to the uniform vector $(1, 1, 1, \dots)/\sqrt{n}$ plus random Gaussian noise of width $O(1/\sqrt{n})$ [12]. Thus the eigenvector centralities of all vertices are $O(1/\sqrt{n})$ with only modest fluctuations. No single node dominates the picture and the eigenvector centrality is well behaved.

If we add the hub to the picture, however, things change. The addition of an extra vertex naturally adds one more eigenvalue and eigenvector to the spectrum, whose values we can calculate by building on methods introduced in [10]. Let \mathbf{X} denote the $(n - 1) \times (n - 1)$ adjacency matrix of the random graph alone and let the vector \mathbf{a} be the first $n - 1$ elements of the final row and

column which represents the hub. (The last element is zero.) Thus the full adjacency matrix has the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{X} & \mathbf{a} \\ \mathbf{a}^T & 0 \end{pmatrix}. \quad (1)$$

Let z be an eigenvalue of \mathbf{A} and let $\mathbf{v} = (\mathbf{v}_1|v_n)$ be the corresponding eigenvector, where \mathbf{v}_1 represents the first $n-1$ elements and v_n is the last element. Then, multiplying out the eigenvector equation $\mathbf{A}\mathbf{v} = z\mathbf{v}$, we find that $\mathbf{X}\mathbf{v}_1 + v_n\mathbf{a} = z\mathbf{v}_1$ and $\mathbf{a}^T\mathbf{v}_1 = zv_n$. Rearranging the first of these, we get

$$\mathbf{v}_1 = v_n(z\mathbf{I} - \mathbf{X})^{-1}\mathbf{a}, \quad (2)$$

and substituting into the second we get

$$\mathbf{a}^T(z\mathbf{I} - \mathbf{X})^{-1}\mathbf{a} = z, \quad (3)$$

where \mathbf{I} is the identity. Replacing the matrix inverse with its eigendecomposition $(z\mathbf{I} - \mathbf{X})^{-1} = \sum_i \mathbf{x}_i(z - \chi_i)^{-1}\mathbf{x}_i^T$, where \mathbf{x}_i is the i th eigenvector of \mathbf{X} and χ_i is the corresponding eigenvalue, Eq. (3) becomes

$$\frac{(\mathbf{a}^T\mathbf{x}_1)^2}{z - (c+1)} + \sum_{i=2}^{n-1} \frac{(\mathbf{a}^T\mathbf{x}_i)^2}{z - \chi_i} = z, \quad (4)$$

where we have explicitly separated the largest eigenvalue $\chi_1 = c+1$ and the remaining $n-2$ eigenvalues, which follow the semicircle law. This equation immediately implies that the eigenvalues z satisfy an *interlacing condition* which says that $n-2$ of the values are unchanged to within $O(1/n)$ from the values χ_i for the network without the hub—we still have the semicircle spectrum plus an outlying eigenvalue $c+1$ —but two new eigenvalues appear, one at the top of the spectrum and one at the bottom. For the purposes of the eigenvector centrality it is the one at the top that we care about, whose value we can calculate from (4) as follows.

The numerator of the first term in Eq. (4) is $O(1/n)$, so this term can be ignored when z is far from $c+1$. Meanwhile \mathbf{x}_i for $i \geq 2$ is uncorrelated with \mathbf{a} and hence the product $\mathbf{a}^T\mathbf{x}_i$ is a Gaussian random variable with variance d/n and, averaging over the randomness, Eq. (4) simplifies to

$$\frac{d}{n} \text{Tr}(z\mathbf{I} - \mathbf{X})^{-1} = z. \quad (5)$$

The quantity $g(z) = n^{-1} \text{Tr}(z\mathbf{I} - \mathbf{X})^{-1}$ is a standard one in the theory of random matrices—it is the so-called Stieltjes transform for a random symmetric matrix, whose value is known to be [12]

$$g(z) = \frac{z - \sqrt{z^2 - 4c}}{2c}. \quad (6)$$

Combining Eqs. (5) and (6) and solving for z we then find the eigenvalue we are looking for:

$$z = \frac{d}{\sqrt{d-c}}. \quad (7)$$

Depending on the degree d of the hub, this eigenvalue may be either smaller or larger than the leading eigenvalue $c+1$ of the random graph. Writing $d/\sqrt{d-c} > c+1$ and rearranging, we see that the hub eigenvalue becomes the leading eigenvalue when

$$d > c(c+1), \quad (8)$$

i.e., when the hub degree is roughly the square of the mean degree. Below this point, the eigenvector centrality is given by the leading eigenvector of the random graph, which, as we have said, is well behaved, so the centrality has no problems. Above this point, however, the leading eigenvector is the one introduced by the hub, and this eigenvector, as we now show, has severe problems.

If the eigenvector $\mathbf{v} = (\mathbf{v}_1|v_n)$ is normalized to unity then $1 = |\mathbf{v}_1|^2 + v_n^2 = v_n^2[\mathbf{a}^T(z\mathbf{I} - \mathbf{X})^{-2}\mathbf{a} + 1]$ from Eq. (2), and hence

$$\begin{aligned} v_n^2 &= \frac{1}{\mathbf{a}^T(z\mathbf{I} - \mathbf{X})^{-2}\mathbf{a} + 1} = \frac{1}{(d/n) \text{Tr}(z\mathbf{I} - \mathbf{X})^{-2} + 1} \\ &= \frac{1}{-dg'(z) + 1}, \end{aligned}$$

where $g(z)$ is again the Stieltjes transform, Eq. (6), and $g'(z)$ is its derivative. Performing the derivative and setting $z = d/\sqrt{d-c}$, we find that

$$v_n^2 = \frac{d-2c}{2d-2c}, \quad (9)$$

which is constant and does not vanish as $n \rightarrow \infty$. In other words, a finite fraction of the weight of the vector is concentrated on the hub vertex.

The neighbors of the hub also receive significant weight: the average of their values is given by

$$\frac{\mathbf{a}^T\mathbf{v}_1}{d} = \frac{v_n}{d}\mathbf{a}^T(z\mathbf{I} - \mathbf{X})^{-1}\mathbf{a} = v_n g(z) = \frac{v_n}{\sqrt{d-c}}. \quad (10)$$

Thus they are smaller than the hub centrality v_n , but still constant for large n . Finally, defining the $(n-1)$ -element uniform vector $\mathbf{1} = (1, 1, 1, \dots)$, the average of all $n-1$ non-hub vector elements is

$$\langle v_i \rangle = \frac{\mathbf{1}^T\mathbf{v}_1}{n-1} = \frac{v_n}{n-1}\mathbf{1}^T(z\mathbf{I} - \mathbf{X})^{-1}\mathbf{a}. \quad (11)$$

Averaging over the randomness and noting that \mathbf{X} and \mathbf{a} are independent and that the average of \mathbf{a} is $d\mathbf{1}/(n-1)$, we then get

$$\langle v_i \rangle = \frac{dv_n}{n-1}g(z) = \frac{1}{n-1} \frac{dv_n}{\sqrt{d-c}}, \quad (12)$$

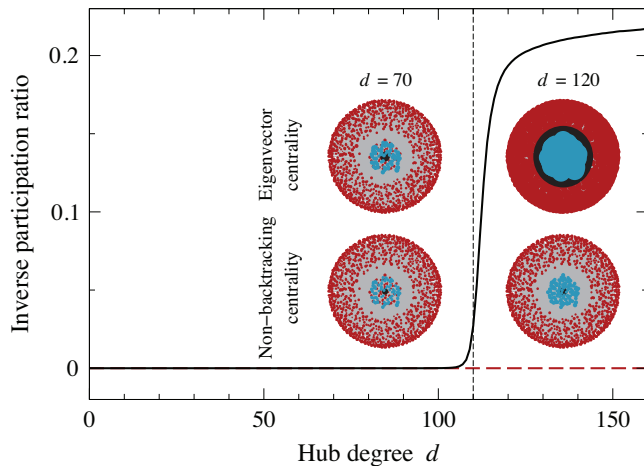


FIG. 1: Main figure: Numerical results for the inverse participation ratio S for the model described in the text, with $n = 1\,000\,000$ vertices and average degree $c = 10$, as a function of hub degree d . The solid curve is for the eigenvector centrality; the dashed curve is the non-backtracking centrality. The vertical dashed line is the expected position $d = 110$ of the localization transition from Eq. (8). Insets: Hub (black), neighbors (blue), and second-neighbors (red) for four examples of the model networks, as described in the text.

which falls off as $1/n$ for large n .

Thus, in the regime above the transition defined by (8), where the eigenvector generated by adding the hub is the leading eigenvector, a non-vanishing fraction of the eigenvector centrality falls on the hub vertex and its neighbors, while the average vertex in the network gets only an $O(1/n)$ vanishing fraction in the limit of large n , much less than the $O(1/\sqrt{n})$ fraction received by all vertices below the transition. This is the phenomenon we refer to as localization: the abrupt focusing of most of the centrality on just a few vertices as the degree of the hub passes above the critical value $c(c+1)$. In the localized regime the eigenvector centrality picks out the hub and its neighbors clearly, but assigns vanishing weight to the average node, and the relative importance of the hub, as measured by the ratio of its centrality to that of the average node, diverges with system size.

As a demonstration of this phenomenon, we show inset in Fig. 1 visualizations of simulated networks generated using our model. Each plot depicts the network neighborhood around the hub in a single million-node network, with the size of the nodes being proportional to their centrality. The top two plots show the situation for the standard eigenvector centrality for two different values of the hub degree— $d = 70$ on the left and $d = 120$ on the right. The former lies well within the regime where there is no localization, while the latter is in the localized regime. The difference between the two is striking—in the first, higher-degree vertices (such as the hub) get higher centrality, as they should, but only modestly so, while in the

second the hub vertex and its neighbors become so large as to dominate the figure.

The extent of the localization can be quantified further by calculating an *inverse participation ratio* $S = \sum_{i=1}^n v_i^4$. In the regime below the transition where there is no localization and all elements v_i are $O(1/\sqrt{n})$ we have $S = O(1/n)$. But if one or more elements are $O(1)$, then $S = O(1)$ also. Hence if there is a localization transition in the network then, in the limit of large n , S will go from being zero to nonzero at the transition in the classic manner of an order parameter. The main plot in Fig. 1 shows just such a transition in our model, falling precisely at the expected position of the localization transition.

So far we have looked only at a simple model of the localization process, but localization occurs in more realistic networks as well. In particular, we expect it to be a problem in networks with a few high-degree hubs or in very sparse networks, those with low average degree c , where it is relatively easy for a hub vertex to exceed the localization threshold. Many real-world networks fall into these categories. Consider, for example, the common case of a network with a power-law degree distribution, such that the fraction p_k of nodes with degree k goes as $k^{-\alpha}$ for some constant exponent α [13]. We can mimic such a network using the so-called configuration model [14, 15], a random graph with specified degree distribution. There are again two different ways a leading eigenvalue can be generated, one due to the average behavior of the entire network and one due to hub vertices of particularly high degree. In the first case the highest eigenvalue for the configuration model is known to be equal to the ratio of the second and first moments of the degree distribution $\langle k^2 \rangle / \langle k \rangle$ in the limit of large network size and average degree [10, 16]. At the same time, the leading eigenvalue must satisfy the Rayleigh bound $z \geq \mathbf{x}^T \mathbf{A} \mathbf{x}$ for any unit-normalized vector \mathbf{x} , with better bounds achieved when \mathbf{x} better approximates the true leading eigenvector. If d denotes the highest degree of any hub in the network and we choose a (normalized) approximate eigenvector of form similar to the one in our model network, having elements

$$x_i = \begin{cases} 1/\sqrt{2} & \text{for the hub,} \\ 1/\sqrt{2d} & \text{for neighbors of the hub,} \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

then it is straightforward to show that the Rayleigh bound implies $z \geq \sqrt{d}$. Thus the eigenvector generated by the hub will be the leading eigenvector whenever $\sqrt{d} > \langle k^2 \rangle / \langle k \rangle$ (possibly sooner, but not later).

In a power-law network with n vertices and exponent α , the highest degree goes as $d \sim n^{1/(\alpha-1)}$ [17] and hence increases with increasing n , while $\langle k^2 \rangle \sim d^{3-\alpha}$ and $\langle k \rangle \sim \text{constant}$ for the common case of $\alpha < 3$. Thus we will have $\sqrt{d} > \langle k^2 \rangle / \langle k \rangle$ for large n provided

$\frac{1}{2} > 3 - \alpha$, so we expect the hub eigenvector to dominate and the eigenvector centrality to fail due to localization when $\alpha > \frac{5}{2}$, something that happens in many real-world networks. (A related point has been made by Chung *et al.* [16], although not in the context of the eigenvector centrality.)

So if eigenvector centrality fails to do its job, what can we do to fix it? Localization occurs because a hub with high eigenvector centrality automatically gives high centrality to its neighbors, which in turn reflect it back again and overinflate the hub's centrality. We can make the centrality well behaved again by preventing this reflection. To achieve this we propose a modified eigenvector centrality, similar in many ways to the standard one, but with an important change. We define the centrality of node j to be the sum of the centralities of its neighbors as before, but the neighbor centralities are now calculated *in the absence of node j* . This is a natural definition in many ways—when I ask my neighbors what their centralities are in order to calculate my own, I want to know their centrality due to their other neighbors, not myself.

This modified eigenvector centrality has the desirable property that when typical degrees are large, so that the exclusion or not of any one node makes little difference, its value will tend to that of the standard eigenvector centrality. But in sparser networks of the kind that tends to give problems, its value will be different from that of the standard measure and, as we will see, better behaved.

Our centrality measure is calculated using the Hashimoto or non-backtracking matrix [6, 7], which is defined as follows. One first converts the undirected network with m edges to a directed one with $2m$ edges by replacing each undirected edge between a pair of vertices with two directed edges pointing in opposite directions. The non-backtracking matrix \mathbf{B} is then the $2m \times 2m$ non-symmetric matrix with one row and one column for each directed edge $i \rightarrow j$ and elements

$$B_{i \rightarrow j, k \rightarrow l} = \delta_{il}(1 - \delta_{jk}), \quad (14)$$

where δ_{ij} is the Kronecker delta. Thus a matrix element is equal to one if edge $i \rightarrow j$ points out of the same vertex that edge $k \rightarrow l$ points into, and edges $i \rightarrow j$ and $k \rightarrow l$ are not pointing in opposite directions between the same pair of vertices, and zero otherwise. Note that, since the non-backtracking matrix is not symmetric, its eigenvalues are in general complex, but the largest eigenvalue is always real, as is the corresponding eigenvector.

The element $v_{i \rightarrow j}$ of the leading eigenvector of the non-backtracking matrix now gives us the centrality of vertex i ignoring any contribution from j , and the full *non-backtracking centrality* x_j of vertex j is defined to be the sum of these centralities over the neighbors of j :

$$x_j = \sum_i A_{ij} v_{i \rightarrow j}. \quad (15)$$

In principle one can calculate this centrality directly by

	Network type	Eigenvector	Non-backtracking
Synthetic	Planted hub, $d = 70$	0.0016	0.0012
	Planted hub, $d = 120$	0.5067	0.0012
	Power law, $\alpha = 2.1$	0.0943	0.0660
	Power law, $\alpha = 2.9$	0.5048	0.0328
Empirical	Physics collaboration	0.0624	0.0623
	Word associations	0.1746	0.0866
	Youtube friendships	0.2188	0.0687
	Company ownership	0.5004	0.1270
	Ph.D. advising	0.5011	0.1964
	Electronic circuit	0.4233	0.0749
	Amazon co-purchasing	0.2257	0.1840

TABLE I: Inverse participation ratio for a variety of networks calculated for traditional eigenvector centrality and the non-backtracking version. The first four networks are computer-generated, as described in the text. The remainder are, in order: a network of coauthorships of papers in high-energy physics [18], word associations from the Free Online Dictionary of Computing [19], friendships between users of the Youtube online video service [20], a network of which companies own which others [21], advisors and advisees in computer science [22], electronic circuit 838 from the ISCAS 89 benchmark set [23], and a product co-purchasing network from the online retailer Amazon.com [18].

calculating the leading eigenvector of \mathbf{B} and then applying Eq. (15). In practice, however, one can perform the calculation faster by making use of the so-called Ihara determinant formula, from which it can be shown that the vector \mathbf{x} of centralities is equal to the first n elements of the leading eigenvector of the $2n \times 2n$ matrix [7]

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{I} - \mathbf{D} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad (16)$$

where \mathbf{A} is the adjacency matrix as previously, \mathbf{I} is the $n \times n$ identity matrix, and \mathbf{D} is the diagonal matrix with the degrees of the vertices along the diagonal. Since \mathbf{M} only has marginally more nonzero elements than the adjacency matrix itself ($2m + 2n$ for a network with m edges and n vertices, versus $2m$ for the adjacency matrix), finding its leading eigenvector takes only slightly longer than the calculation of the ordinary eigenvector centrality.

The bottom two inset networks in Fig. 1 show a visualization of the non-backtracking centrality for the same networks as the top two. Again, vertex size indicates the centrality values and, as the figure makes clear, the measure now remains well behaved all the way up to the previous position of the localization transition and beyond—there is no condensation of the centrality around the hub in the network. The dashed curve in the main figure shows the value of the inverse participation ratio for the non-backtracking centrality and again all evidence of localization has vanished.

The inverse participation ratio also provides a convenient way to test for localization in other networks, both synthetic and real. Table I summarizes results

for eleven networks, for both the traditional eigenvector centrality and the non-backtracking version. The synthetic networks are generated using the random-graph-plus-hub model described here and using the configuration model with power-law degree distribution and in each case there is evidence of localization in the eigenvector centrality in the regimes where it is expected and not otherwise, but no localization at all, in any case, for the non-backtracking centrality. A similar picture is seen in most of the real-world networks—localization in the eigenvector centrality but not the non-backtracking version (e.g., the circuit network) or localization in neither case (e.g., the collaboration network). In at least one network, however (the co-purchasing network), localization appears to persist even when using the non-backtracking centrality. It is unclear what the origin of this behavior is, but it is known that there are other mechanisms by which high-lying eigenvalues can be generated in the spectrum of the adjacency matrix, such as the presence of cliques (completely connected subgraphs), and these too could give rise to localization-like phenomena.

In conclusion, we have shown that the widely used network measure known as eigenvector centrality fails under commonly occurring conditions because of a localization transition in which most of the weight of the centrality concentrates on a small number of vertices. The phenomenon is particularly visible in networks with high-degree hubs or power-law degree distributions, which includes many important real-world examples. We propose a new spectral centrality measure based on the non-backtracking matrix which rectifies the problem in some cases, giving values similar to the standard eigenvector centrality in well behaved cases, but avoiding localization in cases where the standard measure fails. In applications to real-world networks the new measure seems to give significant advantages on some networks, although localization is still visible in some networks. Further work is needed to fully understand the behavior of the measure.

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- [1] P. F. Bonacich, Factoring and weighting approaches to status scores and clique identification. *Journal of Mathematical Sociology* **2**, 113–120 (1972).
 - [2] S. Wasserman and K. Faust, *Social Network Analysis*. Cambridge University Press, Cambridge (1994).
 - [3] S. Brin and L. Page, The anatomy of a large-scale hypertextual Web search engine. *Computer Networks* **30**,

- 107–117 (1998).
- [4] J. M. Kleinberg, Authoritative sources in a hyperlinked environment. *J. ACM* **46**, 604–632 (1999).
- [5] C. T. Bergstrom, J. D. West, and M. A. Wiseman, The eigenfactor metrics. *J. Neurosci.* **28**, 11433–11434 (2008).
- [6] K. Hashimoto, Zeta functions of finite graphs and representations of p-adic groups. *Adv. Stud. Pure Math.* **15**, 211–280 (1989).
- [7] F. Krzakala, C. Moore, E. Mossel, J. Neeman, A. Sly, L. Zdeborová, and P. Zhang, Spectral redemption: Clustering sparse networks. Preprint arXiv:1306.5550 (2013).
- [8] I. J. Farkas, I. Derényi, A.-L. Barabási, and T. Vicsek, Spectra of “real-world” graphs: Beyond the semicircle law. *Phys. Rev. E* **64**, 026704 (2001).
- [9] K. A. Eriksen, I. Simonsen, S. Maslov, and K. Sneppen, Modularity and extreme edges of the Internet. *Phys. Rev. Lett.* **90**, 148701 (2002).
- [10] R. R. Nadakuditi and M. E. J. Newman, Spectra of random graphs with arbitrary expected degrees. *Phys. Rev. E* **87**, 012803 (2013).
- [11] The value of c is allowed to grow modestly with network size $c \sim \log n$ to prevent additional hub nodes from appearing as $n \rightarrow \infty$.
- [12] T. Tao, *Topics in Random Matrix Theory*. American Mathematical Society, Providence, RI (2012).
- [13] A.-L. Barabási and R. Albert, Emergence of scaling in random networks. *Science* **286**, 509–512 (1999).
- [14] M. Molloy and B. Reed, A critical point for random graphs with a given degree sequence. *Random Structures and Algorithms* **6**, 161–179 (1995).
- [15] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Random graphs with arbitrary degree distributions and their applications. *Phys. Rev. E* **64**, 026118 (2001).
- [16] F. Chung, L. Lu, and V. Vu, Spectra of random graphs with given expected degrees. *Proc. Natl. Acad. Sci. USA* **100**, 6313–6318 (2003).
- [17] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Size-dependent degree distribution of a scale-free growing network. *Phys. Rev. E* **63**, 062101 (2001).
- [18] J. Leskovec, J. Kleinberg, and C. Faloutsos, Graph evolution: Densefication and shrinking diameters. *ACM Trans. Knowl. Discov. Data* **1**, 2 (2007).
- [19] V. Batagelj, A. Mrvar, and M. Zaveršnik, Network analysis of dictionaries. In *Proceedings of the 3rd Language Technologies Conference*, Ljubljana, Slovenia (2002).
- [20] A. Mislove, M. Marcon, K. P. Gummadi, P. Druschel, and B. Bhattacharjee, Measurement and analysis of online social networks. In *Proceedings of the 5th ACM/USENIX Internet Measurement Conference (IMC’07)*, San Diego, CA (2007).
- [21] K. Norlen, G. Lucas, M. Gebbie, and J. Chuang, Eva: Extraction, visualization and analysis of the telecommunications and media ownership network. In *Proceedings of the 14th Biennial Conference of the International Telecommunications Society*, Seoul, Korea (2002).
- [22] W. de Nooy, A. Mrvar, and V. Batagelj, *Exploratory Social Network Analysis with Pajek*, Cambridge University Press, Cambridge (2005).
- [23] R. Milo, S. Itzkovitz, N. Kashtan, R. Levitt, S. Shen-Orr, I. Ayzenshtat, M. Sheffer, and U. Alon, Superfamilies of evolved and designed networks. *Science* **303**, 1538–1542 (2004).