

Random matrix analysis of network Laplacians

Sarika Jalan^{*}, Jayendra N. Bandyopadhyay

Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzerstr. 38, D-01187 Dresden, Germany

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Abstract

We analyse the eigenvalue fluctuations of the Laplacian of various networks under the random matrix theory framework. Analyses of random networks, scale-free networks and small-world networks show that the nearest neighbor spacing distribution of the Laplacian of these networks follow Gaussian orthogonal ensemble statistics of the random matrix theory. Furthermore, we study the nearest neighbor spacing distribution as a function of the random connections and find that the transition to the Gaussian orthogonal ensemble statistics occurs at the small-world transition.

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1. Introduction

In order to understand the complex world around us network theory has been getting fast recognition. The main concept here is to define the complex systems in terms of networks of many interacting units. Few examples of such systems are interacting molecules in a living cell, nerve cells in brain, computers in Internet communication, social networks of interacting people, airport networks with flight connections, etc. [1–4]. To understand these networks simple models are introduced; these model networks are based on some simple principles, and capture the essential features of real systems. Mathematically networks are investigated under the framework of graph theory. In the graph theoretical terminology, units are called nodes and interactions are called edges [5].

Random graph model of Erdős and Rényi (ER) assumes that the interaction between the nodes are random [6]. Recently, with the availability of large maps of real-world networks, it has been observed that the random graph model is not appropriate for studying real-world networks. Hence many new models have been introduced. Barabási–Albert’s scale-free (SF) model [7] and Watts–Strogatz’s small-world (SW) model [8] are the most recognized ones, and have contributed immensely in understanding the evolution and behavior of the real systems having network structures. SF model captures the degree distribution behavior of real-world networks and SW model deals with the clustering coefficient and diameter. Following these two new models came an outbreak in the field of networks. These works have focused on the following aspects: (1) direct studies of the real-world networks and measuring their various structural

^{*} Corresponding author. Tel.: +49 351 871 1211.

E-mail addresses: sarika@pks.mpg.de (S. Jalan), jayendra@pks.mpg.de (J.N. Bandyopadhyay).

properties such as degree distribution, diameter, clustering coefficient, etc., (2) proposing new random graph models motivated by these studies, and (3) computer simulations of the new models and measuring their properties [2].

Barabási et al.'s investigations on the various real-world systems show that they are scale-free, which means that the degree distribution $p(k)$, fraction of nodes that have k number of connections with other nodes, decays as power law, i.e. $p(k) \propto k^{-\gamma}$, where γ depends on the topology of the networks. The scale-free nature of the networks suggests that there exist few nodes with very high degree. Some other analyses, by Newman and others, of real-world networks show that complex networks have community or module structures [9,10]. According to these studies, there exists few nodes with very high betweenness which are responsible to connect the different communities. This direction of looking at the networks focuses on the importance of nodes based on its position in the network. On the other hand, ER and SW models emphasize on the random connections in the networks; in ER model any two nodes are connected with probability p . One of the most interesting characteristics of the ER model was the sudden emergence of various global properties; the most important one being the emergence of the giant cluster. For a $p > p_c$, while number of nodes in the graph remain constant, giant cluster emerges through a phase transition. Further, Watts–Strogatz model shows the small-world transition with fine tuning of the number of random connections.

Apart from the above mentioned investigations which focus on direct measurements of structural properties of networks, there exists a vast literature demonstrating that the properties of networks or graphs could be well-characterized by the spectrum of associated adjacency (A) and Laplacian (L) matrix [11]. For an unweighted graph, adjacency matrix is defined in following way : $A_{ij} = 1$, if i and j nodes are connected and *zero* otherwise. Laplacian of the graph has been defined in various ways (depending upon the normalization) in the literature. We follow the definition used in [12];

$$L_{ij} = 1 \quad \text{for } i = j \text{ and } L_{ij} = -A_{ij}/\sqrt{d_i d_j}, \quad (1)$$

where d_i denotes the degree of node i . For undirected networks, adjacency and Laplacian both are symmetric matrices and consequently have real eigenvalues [13]. Eigenvalues of the graph are called graph spectra and they give informations about some basic topological properties of the underlying network. During the last twenty years several important applications of the spectral graph theory in physics and chemistry problems have been discovered [11, 14]. For example liquid flowing through a system of communicating pipes is described by a system of linear differential equations. The corresponding matrix appears to be the Laplacian of the underlying graph. Speed of convergence of the liquid flowing process towards an equilibrium state is measured by the second smallest eigenvalue of L [14]. The second smallest eigenvalue of L is also called the algebraic connectivity of a graph and is used to understand the behavior of dynamical processes on the underlying networks [15]. Particularly, Laplacian spectra of networks have been investigated enormously to understand the synchronization of coupled dynamics on networks [15–17], for example recently extremal eigenvalues of the Laplacian have been shown to have high influence on the synchronizability of the network [18]. Similarly, multiplicities of the eigenvalues, particularly at 0 and 1, have direct relations with the properties of graphs [19,20].

Following our recent works, where we used random matrix theory (RMT) to study the spectral properties of adjacency matrix of various networks [21], in this paper we investigate the spectral properties of Laplacians of networks under RMT framework. Particularly we study the nearest neighbor spacing distribution (NNSD) of the Laplacian matrix of various model networks, namely scale-free, small-world and random networks. We find that in spite of the spectral densities of different model networks being different, their eigenvalue fluctuations are the same and follow the Gaussian orthogonal ensemble (GOE) distribution of RMT. We attribute this universality to the presence of the *similar amount of randomness* in all these networks, and show that randomness in the network connections can be quantified by the Brody parameter coming from RMT. Furthermore, there exists one-to-one correlation between the diameter of the network and the eigenvalue fluctuations of the Laplacian matrix. By changing the number of connections in the network we get a transition to the GOE distribution. As Erdős and Rényi observed that, with the fine tuning of network parameter all nodes get connected with a sudden transition; under the RMT framework our analysis suggests a transition to some kind of spreading of randomness over the whole network.

Note that in this paper we consider the normalized Laplacians, though for the RMT analysis form of the Laplacians does not matter, because unfolding, a method to separate out the system-dependent part from the eigenvalues to study their universal behavior, removes scaling effects caused by the normalization factor. We consider the normalized Laplacians because the eigenvalue properties of this form of the Laplacian are extensively investigated [22–26]. They

have been found to be an excellent candidate as a concise fingerprint of the internet-like graphs [24] and graphs for the cancer cells [25]. Furthermore for some applications, such as graph partitioning, this normalized form is preferred over other forms of the Laplacian [26,20].

The paper is organized as follows: after this introductory section, in Section 2, we describe some techniques of RMT which we have used in our analysis. In Section 3, we analyse the NNSD of the Laplacian for various networks, namely small-world, scale-free and random networks. In Section 4 we study the effect of random connections on the level statistics. In this section we show the one-to-one correlation between the GOE transition and the small-world behavior. Finally, in Section 5, we summarize and discuss about some possible future directions.

2. Random matrix statistics

RMT was proposed by Wigner to explain the statistical properties of nuclear spectra [27]. Later this theory had successfully been applied in the study of different complex systems such as disordered systems, quantum chaotic systems where RMT tells whether corresponding classical system is regular or chaotic or a mixture of both, spectra of large complex atoms, etc. [28]. RMT is also shown to be of great interest in understanding the statistical structure of the empirical cross-correlation matrices appearing in the study of multivariate time series. The classical complex systems where RMT has been successfully applied are stock market (cross-correlation matrix is formed by using the time series of price fluctuations of different stock) [29]; brain (matrix is constructed by using EEG data at different locations) [30]; patterns of atmospheric variability (cross-correlations matrix is generated by using temporal variation of various atmospheric parameters) [31], etc.

We study the eigenvalue fluctuations of the Laplacian of the networks. The eigenvalue fluctuations are generally obtained from the nearest neighbor spacing distribution (NNSD) of the eigenvalues. The NNSD follows two universal properties depending upon the underlying correlations among the eigenvalues. For correlated eigenvalues, the NNSD follows Wigner–Dyson formula of Gaussian orthogonal ensemble (GOE) statistics of RMT; whereas, the NNSD follows Poisson statistics of RMT for uncorrelated eigenvalues.

Here we briefly describe some aspects of RMT which we have used in our network analysis. We denote the eigenvalues of network Laplacian by λ_i , $i = 1, \dots, N$, where N is the size of the network and $\lambda_{i+1} > \lambda_i$, $\forall i$. In order to get the universal properties of the fluctuations of the eigenvalues, it is customary in RMT to unfold the eigenvalues by a transformation $\bar{\lambda}_i = \bar{N}(\lambda_i)$, where \bar{N} is the averaged integrated eigenvalue density [27]. Since we do not have any analytical form for \bar{N} , we numerically unfold the spectrum by polynomial curve fitting. After the unfolding, the average spacing will be *unity*, independent of the system. Using the unfolded spectra, we calculate the spacing as $s_i = \bar{\lambda}_{i+1} - \bar{\lambda}_i$. The NNSD $P(s)$ is defined as the probability distribution of these s_i 's. In the case of Poisson statistics, $P(s) = \exp(-s)$; whereas for GOE

$$P(s) = \frac{\pi}{2} s \exp\left(-\frac{\pi s^2}{4}\right). \quad (2)$$

For the intermediate cases, the spacing distribution is described by the Brody distribution [32]:

$$P_\beta(s) = A s^\beta \exp\left(-\alpha s^{\beta+1}\right), \quad (3)$$

where

$$A = (1 + \beta)\alpha \quad \text{and} \quad \alpha = \left[\Gamma\left(\frac{\beta + 2}{\beta + 1}\right)\right]^{\beta+1}. \quad (4)$$

This is a semiempirical formula characterized by the parameter β . As β goes from 0 to 1, the Brody distribution smoothly changes from Poisson to GOE. We fit the spacing distributions of different networks by the Brody distribution $P_\beta(s)$. This fitting gives an estimation of β , and consequently identifies whether the spacing distribution of a given network is Poisson or GOE or intermediate of these *two*.

3. Laplacian matrix spectrum of complex networks

In the following we present the ensemble averaged spectral density and spacing distribution of random, scale-free and small-world networks.

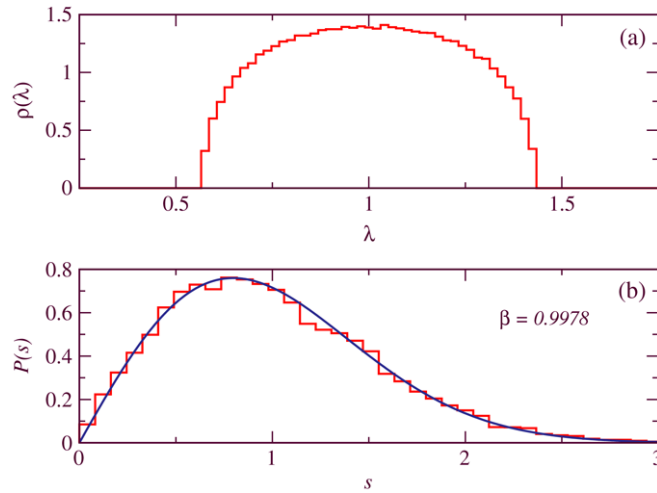


Fig. 1. (Color online) (a) and (b) show the spectral density ($\rho(\lambda)$) and corresponding spacing distribution ($P(s)$) for the random network. The histograms are numerical results and the solid lines represent fitted Brody distribution. Figures are plotted for an average over 10 random realizations of the networks. All networks have $N = 2000$ nodes and an average degree $k = 20$ per node.

3.1. Random network

First we consider the random network generated by the using Erdős and Rényi algorithm. We take $N = 2000$ nodes and make random connections between pairs of nodes with probability $p = 0.01$. This method yields a connected network with an average degree $p \times N = 20$. Note that for a very small value of p one gets several unconnected components. Here, the choice of p is such that it should be high enough ($p > p_c$) to give a large connected component typically spanning all nodes [33]; and since most of the real-world networks are very sparse [2], p is small enough to have a sparse network. Laplacian is constructed using Eq. (1). Because of the normalized form, the eigenvalues would always be within 0 and 2. Fig. 1(a) plots the spectral density of the Laplacian. Note that the distribution is averaged over 10 realizations of the network. The spectral density follows Wigner–Dyson semicircular distribution [34]. To get the spacing behaviors, first the eigenvalues are unfolded by using the technique described in Section 2. This method yields the eigenvalues with constant spectral density on the average. These unfolded eigenvalues are used to calculate NNSD. The same procedure is repeated for an ensemble of the networks generated for different random realizations. Fig. 1(b) plots the ensemble average of NNSD. By fitting this ensemble averaged NNSD with the Brody formula given in Eq. (3) we get an estimation of the Brody parameter $\beta = 0.9786 \sim 1$. This value of Brody parameter clearly indicates the GOE behavior of the NNSD (Eq. (2)).

3.2. Scale-free network

Fig. 2 plots the spectral density and the nearest neighbor spacing distribution of the Laplacian of scale-free network. Scale-free network is generated by using the model of Barabási et al. [7]. Starting with a small number, m_0 of the nodes, a new node with $m \leq m_0$ connections is added at each time step. This new node connects node i with the probability $\pi(k_i) \propto k_i$ (preferential attachment), where k_i is the degree of the node i . After τ time steps the model leads to a network with $N = \tau + m_0$ nodes and $m\tau$ connections. This model leads to a scale-free network, i.e., the probability $P(k)$ that a node has degree k decays as a power law, $P(k) \sim k^{-\lambda}$, where λ is a constant and for the type of probability law $\pi(k)$ that we have used $\lambda = 3$. Other forms for the probability $\pi(k)$ are possible which give different values of λ and we find results similar to the ones reported here. Density distribution of the network has a very uniform distribution for almost all eigenvalues except for few lowest and highest ones. There is a slight peak around one which corresponds to the peak at zero for the adjacency matrix [21]. To calculate NNSD, we follow the same procedure as described in the previous section. Fig. 2(b) shows that the spacing distribution of this scale-free network follow GOE very closely ($\beta = 0.9914 \simeq 1$).

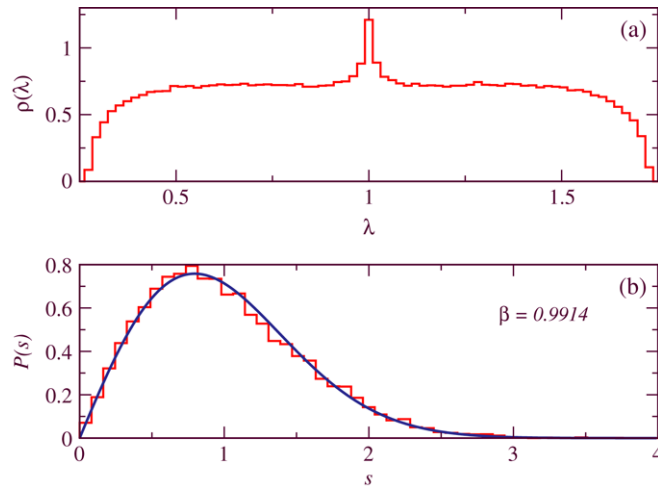


Fig. 2. (Color online) (a) and (b) show the spectral density ($\rho(\lambda)$) and corresponding spacing distribution ($P(s)$) of scale-free network. The histograms are numerical results and the solid lines represent fitted Brody distribution. Figures are plotted for an average over 10 random realizations of the networks. All networks have $N = 2000$ nodes and an average degree $k = 20$ per node.

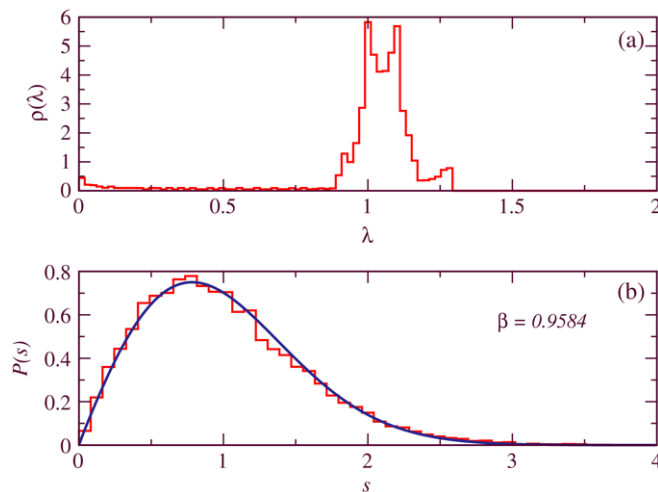


Fig. 3. (Color online) (a) and (b) show the spectral density ($\rho(\lambda)$) and corresponding spacing distribution ($P(s)$) of small-world network. The histograms are numerical results and the solid lines represent fitted Brody distribution. Figures are plotted for an average over 10 random realizations of the networks. All networks have $N = 2000$ nodes and an average degree $k = 20$ per node.

3.3. Small-world network

Fig. 3 is plotted for small-world network. Small-world networks are constructed using the following algorithm by Watts and Strogatz [8]. Starting with a one-dimension ring lattice of N nodes in which every node is connected to its k nearest neighbors, we randomly rewire each connection of the lattice with the probability p such that the self-loop and multiple connections are excluded. Thus $p = 0$ gives a regular network and $p = 1$ gives a random network. The typical small-world behavior is observed around $p = 0.005$. We take $N = 2000$ and average degree $k = 40$. Spectral density of this network is complicated with several peaks. One peak is at $\lambda = 1$ which corresponds to the $\rho(0)$ peak for the spectra of the adjacency matrix. For different values of k the exact positions of other peaks may vary but the overall form of the spectral density remains similar. Fig. 3(b) shows that the spacing distribution of small-world network follows GOE very closely ($\beta = 0.9584 \simeq 1$).

Figs. 1(b), 2(b) and 3(b) show that the spacing distribution of all the three networks follow GOE very closely ($\beta \simeq 1$). Since the spectral density of the random network is very close to the semicircular one, and in the RMT

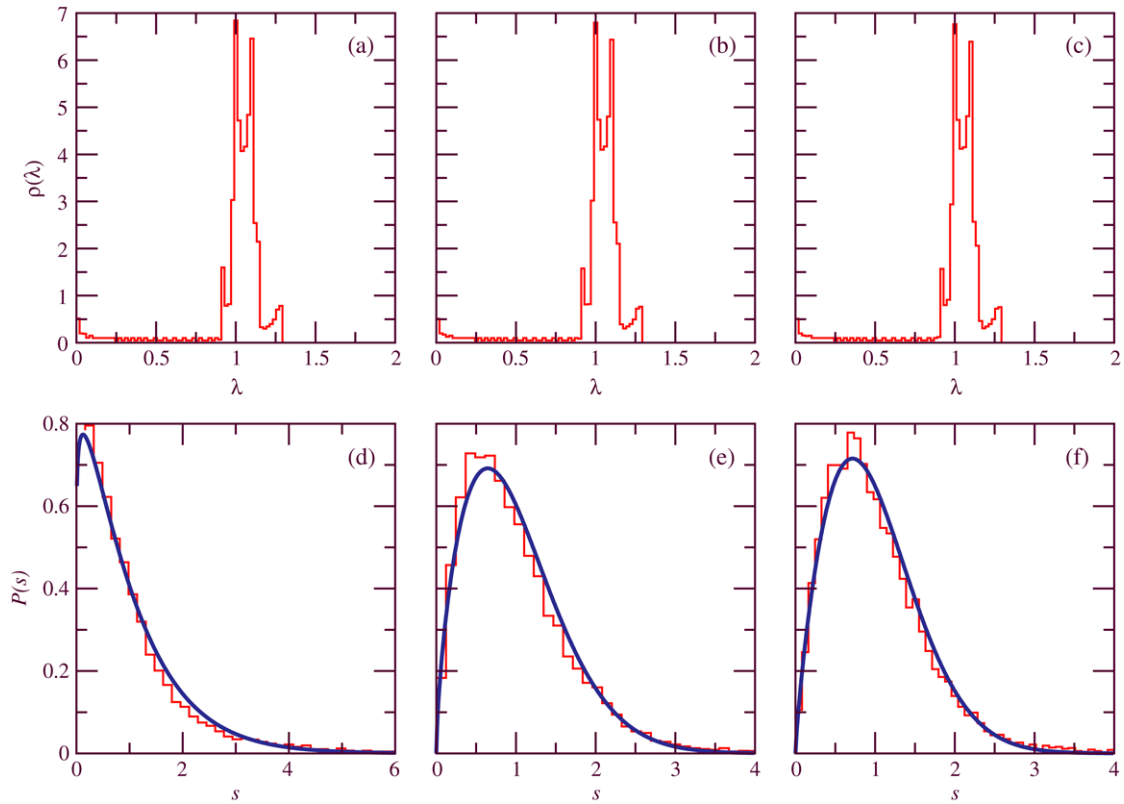


Fig. 4. Figures shows the transition from the Poisson to the GOE statistics of the eigenvalues of the Laplacian of the network as a function of the random rewiring parameter p . We take a regular lattice of size $N = 2000$ and average degree $k = 40$. Links are rewires with probability p . Figures are plotted for an average over 10 random realization of the networks.

literature semicircular distributions are extensively studied and are shown to give GOE statistics of the spacings, we expected that the NNSD of random networks would follow GOE as well. But for the scale-free network and the small-world network, density distributions have very different forms, still the NNSD of these networks follow GOE statistics. This is a very interesting result. Following RMT these results imply that even though the spectral density of the scale-free and small-world network differ from the spectral density of the random network, but the correlations among the eigenvalues are strong enough to yield GOE statistics.

4. Transition to GOE statistics and small-world behavior

All the above networks have some amount of random connections in them. In this section we discuss the networks which are regular and analyse the spectral properties as we go from regular network to the random one. Starting with a ring lattice of nodes $N = 2000$ and average degree $k = 40$, we then randomize some connections with probability p , and observe the change in the spectral density and NNSD of the network as a function of p . For the original ring lattice, the underlying adjacency matrix would be a band matrix with entries one in the band, except diagonal elements which would be zero. Corresponding Laplacian would be a band matrix as well, with off-diagonal elements $-1/\sqrt{k \times k}$ (from Eq. (1)), where k being the degree of each node, and diagonal elements 1. As a result of rewiring, few randomly selected pairs of nodes get connected, yielding few nonzero entries outside the band of the corresponding Laplacian, while making few band elements zero. For a completely random network, the corresponding Laplacian would have nonzero random entries.

In the following we discuss the spectral behavior of the Laplacian matrix as the number of random connections in the underlying network is increased. Fig. 4(a) and (d) plot the density distribution and the NNSD for the $p = 5 \times 10^{-5}$ case. At this value of p there is as few as one random rewiring in the regular lattice. These figures shows that spectral density of the lattice is complicated without having any known form; and its spacing distribution closely follows

Table 1

Diameter, clustering coefficient, and Brody parameter for the various values of rewiring probability p

	p	$L(p)/L(0)$	$C(p)/C(0)$	$1 - \beta$
(a)	5×10^{-5}	0.83925	0.99980	0.89463
(b)	2×10^{-4}	0.53680	0.99925	0.33368
(c)	5×10^{-4}	0.39776	0.99857	0.20131

the Poisson statistics ($\beta \sim 0.1$). We then increase the value of p and randomize a fraction $p = 2 \times 10^{-4}$ of the edges. For this value of p , the spectral density and the spacing distribution are plotted respectively in Fig. 4(b) and (e). These figures reveal that for this small value of p , the density distribution of the network does not show any noticeable change, whereas the spacing distribution shows a different property (corresponding to the $\beta \sim 0.67$). As p is further increased to 5×10^{-4} the spectral density hardly shows any change but interestingly the spacing distribution shows significantly different properties than the Poisson statistics. Now the spacing distribution looks more like the intermediate between Poisson and GOE. By fitting the spacing distribution corresponding to this p value with the Brody formula, we estimate the Brody parameter β as $\beta \sim 0.8$. As a marked change in the Brody parameter for very small change in the random connections in the network (corresponding to $p = 5 \times 10^{-5}$, Fig. 4(d)), we had expected that the Brody parameter will reach asymptotically to unity as we increase p . But the surprising finding is that the onset of transition from Poisson to GOE occurs at a very small value of p (Fig. 4(e) and (f)), and is related with the SW properties of the networks. We calculate the network diameters and clustering coefficients for these values of p , which are listed below:

Table 1 shows that for (b) and (c), the average length of the network is as small as the corresponding random network ($L_{\text{random}} \sim \ln(N)/\ln(k)$), and clustering coefficient is as high as the regular lattice ($C_{\text{regular}} \sim 3/4$), which are the properties of small-world networks. Values of β reveal that the Poisson to GOE ($\beta \sim 0 \rightarrow \beta \sim 1$) transition and small-world transition takes place for the similar value of p .

5. Conclusions

In summary, we study the eigenvalue spacing distribution of the Laplacian of the model networks studied extensively in the literature. They follow the universal GOE statistics. From the RMT analogy it tells us that there exists correlations between the eigenvalues of the network arising from certain symmetries in the interactions. We attribute this universality to the *similar* amount of randomness in the model networks, which may arise naturally, in order to capture the real-world network properties. We study the effect of the randomness in the network connections on the eigenvalues fluctuations of the network Laplacian, and use the Brody parameter to quantify this randomness. These studies reveal that there is a direct relation between the random connections and the Brody parameter. For the regular network (with the average degree greater than some value) we get the Poisson distribution, as we make random rewiring of the connections the spacing distribution starts deviating from the Poisson statistics, first it shows the intermediate of the Poisson and the GOE statistics and for *sufficiently large* number of random connections it show almost GOE statistics ($\beta \sim 1$). According to the interpretation of RMT, at this value of p , eigenvalues are as much correlated as for the completely random networks. Furthermore we find that this transition to GOE statistics happens at the onset of the small-world behavior.

Universal RMT results shown by the networks suggest that they can be modeled as a random matrix chosen from the GOE of RMT. Eigenvalue fluctuations following GOE statistics argue that there exists some kind of spreading over the randomness in the whole network, which may be essential for the robustness of the system. According to many recent studies, randomness in the connections is one of the most important and desirable ingredients for the proper functionality or the efficient performance of the system having underlying network structure. For instance, information processing in the brain is considered to be highly influenced by random connections among different modular structures [35]. Our analysis suggests that *randomness* in the complex networks can be studied under the RMT framework. Furthermore, the Laplacian spectra have been investigated to understand the various dynamical processes on networks [16–18] as well, hence the RMT analysis of the network Laplacians could also be important in view of these demonstrations of the relations between spectral properties and dynamical processes.

Following the RMT analysis of the adjacency matrices of networks introduced in our previous work [21], we investigate the random matrix properties of network Laplacians. In this paper we only concentrate on the spacing

distribution of the eigenvalues of the Laplacian, which explains short-range correlations among the eigenvalues. Further investigations would involve more sensitive analysis like Δ_3 statistics to understand the long-range correlations [36].

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References

- [1] S.H. Strogatz, *Nature* 410 (2001) 268.
- [2] R. Albert, A.-L. Barabási, *Rev. Modern Phys.* 74 (2002) 47 and reference therein.
- [3] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, D.-U. Hwang, *Phys. Rep.* 424 (2006) 175.
- [4] L.da.F. Costa, F.A. Rodrigues, G. Travieso, P. Villas Boas, *Adv. Phys.* 56 (2007) 167.
- [5] Béla Bollobás, *Random Graphs*, second ed., Cambridge Univ. Press, 2001.
- [6] P. Erdős, A. Rényi, *Publ. Math. Inst. Hungar. Acad. Sci.* 5 (1960) 17.
- [7] A.-L. Barabási, R. Albert, *Science* 286 (1999) 509.
- [8] D.J. Watts, S.H. Strogatz, *Nature* 440 (1998) 393.
- [9] M. Girvan, M.E.J. Newman, *Proc. Natl. Acad. Sci. USA* 99 (2002) 7821–7826;
A. Clauset, M.E.J. Newman, C. Moore, *Phys. Rev. E* 70 (2004) 066111;
M.E.J. Newman, *Soc. Netw.* 27 (2005) 39–54;
M.E.J. Newman, *Proc. Natl. Acad. Sci. USA* 103 (2006) 8577–8582.
- [10] R. Guimerá, L.A.N. Amaral, *Nature* 433 (2005) 895.
- [11] D.M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs: Theory and Applications*, 3rd revised ed., Academic Press, 1997.
- [12] F.R.K. Chung, *Spectral Graph Theory*, Number 92, AMS, 1997.
- [13] Relation between the spectra of L and A is known only in the case of d -regular (degree of each node of the network being equal, $d_i = d \forall i$) graphs. In this very special case, $L = I - A/d$, where I is a unit matrix, and hence $\lambda_L = 1 - \lambda_A/d$, where λ_L and λ_A are the eigenvalues spectra of L and A respectively. For this case, L and A are linearly related, and therefore the statistical properties of their spectra would be same. If the degree of nodes are not same, as usually happens in complex networks, then relation between the eigenvalues of L and A is not known.
- [14] M. Doob, in: J.L. Gross, J. Yellen (Eds.), *Handbook of Graph Theory*, Chapman & Hall/CRC, 2004.
- [15] L.M. Pecora, T.L. Carroll, *Phys. Rev. Lett.* 80 (1998) 2109;
A.E. Motter, Y.-C. Lai, F.C. Hopfsteadt, *Phys. Rev. Lett.* 91 (2003) 014010;
C. Zhou, A.E. Motter, J. Kurths, *Phys. Rev. Lett.* 96 (2006) 034101.
- [16] F.M. Atay, T. Biyikoglu, J. Jost, *IEEE Trans. Circuits Syst.* 1 53 (2006) 92; *Physica D* 224 (2006) 35.
- [17] P.N. McGraw, M. Menzinger, *Phys. Rev. E* 75 (2007) 027104.
- [18] D.-H. Kim, A.D. Motter, *Phys. Rev. Lett.* 98 (2007) 248701.
- [19] R. Grone, R. Merris, V.S. Sunder, *SIAM J. Matrix Anal. Appl.* 11 (1990) 218.
- [20] A. Banerjee, J. Jost, *Theory Biosci.* 126 (2007) 15.
- [21] J.N. Bandyopadhyay, S. Jalan, *Phys. Rev. E* 76 (2007) 026109;
S. Jalan, J.N. Bandyopadhyay, *Phys. Rev. E* 76 (2007) 046107.
- [22] F. Chung, R.M. Richardson, in: B. Berkolaiko, et al. (Eds.), *Quantum Graphs and their Applications*, in: *Contemporary Math.*, AMS, Providence, RI, 2006.
- [23] P. Zhu, R. Wilson, *Lecture Notes in Computer Science*, vol. 3691, Springer, Berlin, 2005, p. 371;
D. Kim, B. Kahng, [eprint:arXiv:cond-mat/0703055](http://arxiv.org/abs/cond-mat/0703055).
- [24] D. Vukadinovi, P. Huang, T. Erlebach, *Lecture Notes in Computer Science*, vol. 234, Springer, Berlin, 2002, p. 83.
- [25] C. Demir, S.H. Gultekin, B. Yener, in: *Proc. of 4th Conference on Modeling and Simulation in Biology, Medicine and Biomedical Engineering*, Linköping, Sweden, May 2005.
- [26] J. Huang, MPI-KYB Technical Report No. 144. Available at: <http://www.cs.uwaterloo.ca/~j9huang/Research/comview.pdf>.
- [27] M.L. Mehta, *Random Matrices*, second ed., Academic Press, New York, 1991.
- [28] T. Guhr, A. Muller-Groeling, H.A. Weidenmuller, *Phys. Rep.* 299 (1998) 189.
- [29] V. Pleron, P. Gopikrishnan, B. Rosenow, L.A.N. Amaral, H.E. Stanley, *Phys. Rev. Lett.* 83 (1999) 1471.
- [30] P. Seba, *Phys. Rev. Lett.* 91 (2003) 198104.
- [31] M.S. Santhanam, P.K. Patra, *Phys. Rev. E* 64 (2001) 016102.
- [32] T.A. Brody, *Lett. Nuovo Cimento* 7 (1973) 482.
- [33] Before this critical value of $p \leq p_c$, one gets several disconnected components. Laplacian matrix L of a network having several disconnected components can be written as Kronecker or direct sum of the Laplacian matrices L_i of the disconnected components, i.e., $L = \bigoplus_{i=1}^N L_i$, where N is the number of disconnected components. In this case, even if the NNSD of the individual components follows GOE, but in general the NNSD of L does not follow GOE property.
- [34] F. Chung, L. Lu, V. Vu, *Proc. Natl. Acad. Sci.* 100 (2003) 6313.
- [35] J.D. Cohen, F. Tong, *Science* 293 (2001) 2405.
- [36] S. Jalan, J.N. Bandyopadhyay, M.S. Santhanam, (in preparation).