

Frozen variables in random boolean constraint satisfaction problems

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Abstract

We determine the exact *freezing threshold*, r^f , for a family of models of random boolean constraint satisfaction problems, including NAE-SAT and hypergraph 2-colouring, when the constraint size is sufficiently large. If the constraint-density of a random CSP, F , in our family is greater than r^f then for almost every solution of F , a linear number of variables are *frozen*, meaning that their colours cannot be changed by a sequence of alterations in which we change $o(n)$ variables at a time, always switching to another solution. If the constraint-density is less than r^f , then almost every solution has $o(n)$ frozen variables.

Freezing is a key part of the clustering phenomenon that is hypothesized by non-rigorous techniques from statistical physics. The understanding of clustering has led to the development of advanced heuristics such as Survey Propagation. It has been suggested that the freezing threshold is a precise algorithmic barrier: that for densities below r^f the random CSPs can be solved using very simple algorithms, while for densities above r^f one requires more sophisticated techniques in order to deal with frozen clusters.

1 Introduction

The clustering phenomenon is arguably the most important development in the study of random constraint satisfaction problems (CSP's) over the past decade or so. Statistical physicists have discovered that for typical models of random constraint satisfaction problems, the structure of the solution space appears to undergo remarkable changes as the constraint density increases.

A common geometric interpretation of the clustering analysis paints the following picture. Most of it is not proven rigorously; in fact many details are not specified precisely. Nevertheless, there is evidence that something close to this takes place for many natural random CSP's: At first, all solutions are very similar in that we

can change any one solution into any other solution via a sequence of small local changes; i.e. by changing only $o(n)$ variables-at-a-time, always having a satisfying solution. This remains true for almost all solutions until the *clustering threshold* [42, 43], at which point they shatter into an exponential number of clusters. Roughly speaking: one can move from any solution to any other *in the same cluster* making small local changes, but moving from one cluster to another requires changing a linear number of variables in at least one step. As we increase the density further, we reach the *freezing threshold* [51]. Above that point, almost all clusters¹ contain *frozen variables*; that is, variables whose values do not change for any solutions in the cluster. At higher densities, we find other thresholds, such as the *condensation threshold* [36] above which the largest cluster contains a positive proportion of the solutions. Eventually we reach the *satisfiability threshold*, the point at which there are no solutions.

The methods that are used to describe these phenomena and determine the values of the thresholds are mathematically sophisticated, but are typically not rigorous. Nevertheless, they have transformed the rigorous study of random CSP's.

For one thing, this picture explained things that mathematicians had already discovered. For some problems (eg. k -NAE-SAT [10], k -SAT [12] and k -COL [11]) the second moment method had been used to prove the existence of solutions at densities that are close to, but not quite, the hypothesized satisfiability threshold. We now understand that this is because the way that the second moment method was applied cannot work past the condensation threshold. As another example, it had long been observed that at a point where the density is still far below the satisfiability threshold, no algorithms are proven to find solutions for many of the standard random CSP models. We now understand [43] that this observed “algorithmic barrier” is asymptotically equal to the clustering threshold as k grows ([3] provides rigorous grounding for this), and

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¹By this we mean: all but a vanishing proportion of the clusters, when weighted by their size.

so the difficulties appear to arise from the onset of clusters. It has been suggested that this algorithmic barrier may occur precisely at the freezing threshold; i.e. the formation of clusters does not cause substantial algorithmic difficulties until most of the clusters have frozen variables (see section 1.1 below).

Although the picture described above is, for the most part, not established rigorously, understanding it has led to substantial new theorems [20, 39, 52, 29, 48, 23, 21, 22, 28, 1, 33, 5, 6]. For example, [23] used our understanding of how condensation has foiled previous second moment arguments to modify those arguments and obtain a remarkably tight bound on the satisfiability threshold for k -NAE-SAT. [21] used our understanding of clustering to design an algorithm that provably solves random k -SAT up to densities of $O(2^k \ln k/k)$, which is the asymptotic value of the clustering threshold. A particularly impressive heuristic result is the Survey Propagation algorithm [43, 16], which experimentally has solved random 3-SAT on 10^7 variables at densities far closer to the satisfiability threshold than anyone had previously been able to handle, even on fewer than 1000 variables. This algorithm was designed specifically to take advantage of the clustering picture.

Of course, another thrust has been to try to rigorously establish pieces of the clustering picture [3, 4, 48, 8, 30, 24, 54, 46]. We have been most successful with k -XOR-SAT; i.e. a random system of boolean linear equations. The satisfiability threshold was established in [27] for $k = 3$ and in [26] for $k \geq 4$. More recently, [8, 30] each established a very precise description of the clustering picture. It should be noted that the solutions of a system of linear equations are very well-understood, and that was of tremendous help in the study of the clustering of the solutions. Other CSP's, for which we do not have nearly as much control over the solutions, have been much more resistant to rigorous analysis; nevertheless, there have been substantial results - see Section 1.2.

The contribution of this paper is to rigorously determine the precise freezing threshold for a family of CSP models including k -NAE-SAT and hypergraph 2-colouring. The freezing threshold for k -COL was determined by the first author in [46]; prior to this work, k -COL and k -XOR-SAT are the only two common models for which the freezing threshold was determined rigorously.

We follow the approach of [46], but we differ mainly in: (i) Where [46] analyzed the *Kempe-core*, we need to analyze the **-core*, which was introduced in [13] to prove the existence of frozen variables in random k -SAT. (ii) Rather than carrying out the analysis for a single model, we carry it out simultaneously for a family of models.

Our informal description of freezing described it in terms of the clusters. At this point, not enough information about clustering has been established rigorously to permit us to define freezing in those terms. (Eg. we do not know the exact clustering threshold for any interesting model except k -XOR-SAT.) So our formal definition of a frozen variable avoids the notion of clustering.

DEFINITION 1.1. *An ℓ -path of solutions of a CSP F is a sequence $\sigma_0, \sigma_1, \dots, \sigma_t$ of solutions, where for each $0 \leq i \leq t-1$, σ_i and σ_{i+1} differ on at most ℓ variables.*

DEFINITION 1.2. *Given a solution σ of a CSP F , we say that a variable x is ℓ -frozen with respect to σ if for every ℓ -path $\sigma = \sigma_0, \sigma_1, \dots, \sigma_t$ of solutions of F , we have $\sigma_t(x) = \sigma(x)$.*

In other words, it is not possible to change the value of v by changing at most ℓ vertices at a time. Roughly speaking, the solutions in the same cluster as σ are the solutions that can be reached by a $o(n)$ -path. So x is $o(n)$ -frozen with respect to σ if x has the same value in every solution in the same cluster as σ . Thus, this definition is essentially equivalent to the informal one if the clustering picture is accurate.

We make critical use of the *planted model* (section 3); [3] permits us to do so. We prove that one can use the planted model up to a certain density, and so we want the freezing threshold to be below that density. It will be if the constraint size k is sufficiently large; $k \geq 30$ will do.

We analyze CSP-models satisfying certain properties: non-trivial, feasible, symmetric, balance-dominated, and 1-essential (defined in section 2). The first four are needed to permit the planted model; the fifth allows us to focus on the **-core*. Given such a CSP model Υ , we define constants $r_f(\Upsilon), r_p(\Upsilon)$ and function $\lambda(\Upsilon, r)$ below. Our main theorem is that $r_f(\Upsilon)$ is the freezing threshold for Υ and that $\lambda(\Upsilon, r)$ is the proportion of frozen vertices. We require the density to be below $r_p(\Upsilon)$ in order to apply the planted model. This is not just a technicality - if the density is significantly above $r_p(\Upsilon)$, then it will be above the condensation threshold and the expressions that we provide will fail to yield the correct constants.

Given a CSP-model Υ , $C(\Upsilon, n, M)$ is a random instance of Υ on n variables and with M constraints (see Section 2). We say that a property holds w.h.p. (with high probability) if it holds with probability tending to 1 as $n \rightarrow \infty$.

THEOREM 1.1. *Consider any non-trivial, feasible, symmetric, balance-dominated, and 1-essential CSP-model Υ with $r_f(\Upsilon) < r_p(\Upsilon)$. Let σ be a uniformly random solution of $C(\Upsilon, n, M = rn)$.*

- (a) For any $r_f(\Upsilon) < r < r_p(\Upsilon)$, there exists a constant $0 < \beta < 1$ for which:
- (i) w.h.p. there are $\lambda(\Upsilon, r)n + o(n)$ variables that are βn -frozen with respect to σ .
 - (ii) w.h.p. there are $(1 - \lambda(\Upsilon, r))n + o(n)$ variables that are not 1-frozen with respect to σ .
- (b) For any $r < r_f(\Upsilon)$, w.h.p. at most $o(n)$ variables are 1-frozen with respect to σ .

In other words, in a typical solution: for $r > r_f$, a linear number of variables are αn -frozen, while for $r < r_f$, all but at most $o(n)$ variables are not even 1-frozen. Furthermore, for $r > r_f$ we specify the number of αn -frozen vertices, up to an additive $o(n)$ term. All but at most $o(n)$ of the other vertices are not even 1-frozen.

We remark that for k -COL and k -XOR-SAT, we have “ $\omega(n)$ -frozen” rather than “1-frozen”, for some $\omega(n) \rightarrow \infty$. Part (b) probably remains true upon replacing “ $o(n)$ ” with “zero”. The $o(n)$ terms arises from a limitation of using the planted model.

For $k \geq 30$ we always have $r_f(\Upsilon) < r_p(\Upsilon)$ (see Proposition 9.1) and so our theorem applies.

For densities below the freezing threshold, our proof yields that, in fact, almost all variables can be changed via a $o(n)$ -path of length 1:

THEOREM 1.2. *Consider any non-trivial, feasible, symmetric, balance-dominated, and 1-essential CSP-model Υ with $r_f(\Upsilon) < r_p(\Upsilon)$. Let σ be a uniformly random solution of $C(\Upsilon, n, M = rn)$ with $r < r_f(\Upsilon)$.*

For any $\omega(n) \rightarrow \infty$, w.h.p. for all but at most $o(n)$ variables x , there is a solution σ' such that (i) $\sigma'(x) \neq \sigma(x)$ and (ii) $\sigma'(x), \sigma(x)$ differ on at most $\omega(n)$ variables.

As mentioned above, our theorems apply to k -NAE-SAT and hypergraph 2-colouring, two of the standard benchmark models. k -NAE-SAT is a k -CNF boolean formula which is satisfied if every clause contains at least one true literal and at least one false literal. For hypergraph 2-colouring, we are presented with a k -uniform hypergraph and we need to find a boolean assignment to the vertices so that no hyperedge contains only vertices of one sign. Thus, it is equivalent to an instance of k -NAE-SAT where every literal is signed positively. See Appendix 8 for a discussion of other models to which our theorems apply.

Physicists tell us that there is a second freezing threshold, above which *every* solution has frozen variables [51, 53] (as opposed to *almost every* solution as in Theorem 1.1). [13] proves that this occurs in k -SAT

for large enough densities (albeit for a weaker notion of freezing); see Section 1.2. We do not see how to determine the exact value of that threshold.

We should emphasize that the clustering picture described above is very rough. The mathematical analysis used by statistical physicists to determine the various thresholds actually studies properties of certain Gibbs distributions on infinite trees rather than solutions of random CSP's. The clustering picture is a common geometric interpretation and it is not exact. Nevertheless, there is very strong evidence that something close to this picture should hold.

1.1 The algorithmic barrier A great deal of the interest in random CSP's arises from the long-established observation that as the densities approach the satisfiability threshold, the problems appear to be extremely difficult to solve [18, 44]. Much work has gone into trying to understand what exactly causes dense problems to be so algorithmically challenging (eg. [19, 2, 21, 43, 47]).

It has been suggested (eg. [55, 53, 34, 35, 25, 51]) that, for typical CSP's, the freezing threshold forms an algorithmic barrier. For $r < r_f$ very simple algorithms (eg. greedy algorithms with minor backtracking steps) will w.h.p. find a satisfying solution, but for $r > r_f$ one requires much more sophisticated algorithms (eg. Survey Propagation). It has been proposed that the following simple algorithm should succeed for $r < r^f$:

Suppose that Theorem 1.2 were to hold for *every* solution σ . We build our CSP one random constraint at a time, letting F_i denote the CSP with i constraints. We begin with a solution σ_0 for F_0 (σ_0 can be any assignment). Then we obtain σ_{i+1} from σ_i as follows: If σ_i does not violate the $(i+1)$ st constraint added, then we keep $\sigma_{i+1} = \sigma_i$. Otherwise, we modify σ_i into another solution σ' of F_i in which the values of the variables in the $(i+1)$ st constraint are changed so that it is satisfied; then we set $\sigma_{i+1} = \sigma'$. If Theorem 1.2 holds for σ_i , then we can change each of the k variables in that constraint by changing only $o(n)$ other variables. Expansion properties of a random CSP imply that these small changes will (usually) not interfere with each other and so we can change each of the k variables to whatever we want. Thus we will eventually end up with a solution σ_M to our random CSP F_M .

However, Theorem 1.2 does not hold for *every* solution, only most of them. This is not just a limit of our proof techniques - it is believed that it does not hold for an exponentially small, but positive, proportion of the solutions. So proving that this algorithm works would require showing that we never encounter one of those solutions.

To see, intuitively, why the onset of freezing may create algorithmic difficulties, consider *near-solutions* - assignments which violate only a small number of constraints, say $o(n)$ of them. The near-solutions will also form clusters (because of *high energy barriers*; see [3]). Furthermore, almost all clusters of near-solutions will not contain any solutions. This is because, above the freezing threshold, almost all solution clusters have a linear number of frozen variables and so after adding only $o(n)$ constraints, we will pick a constraint that violates the frozen variables. This will violate all solutions in that cluster, thus forming a near-solution cluster that contains no actual solutions. Of course, this description is non-rigorous but it provides a good intuition.

Now consider a greedy algorithm with backtracking. As it sets its variables, it will approach a near-solution ρ . At that point, it cannot move to a near-solution in a different cluster than ρ , without employing a backtracking step that changes a linear number of variables. So the algorithm will need to be sophisticated enough to approach one of the rare near-solution clusters that contains solutions.

As described above, there is a second freezing threshold, above which *every* cluster has frozen variables. [55] suggests that this is another algorithmic barrier above which even the sophisticated algorithms fail to find solutions. One indication is that, empirically, every solution σ found by Survey Propagation is such that no variables are frozen with respect to σ . So somehow, the algorithm is drawn to those rare unfrozen clusters, and hence may fail when there are no such clusters.

1.2 Related work The clustering picture for k -NAE-SAT and hypergraph 2-colouring was analyzed non-rigorously in [25]. There are hundreds of other papers from the statistical physics community analyzing clustering and related matters. Some are listed above; rather than listing more, we refer the reader to the book [41].

Achlioptas and Ricci-Tersenghi [13] were the first to rigorously prove that freezing occurs in a random CSP. They studied random k -SAT and showed that for $k \geq 8$, for a wide range of edge-densities below the satisfiability threshold and for *every* satisfying assignment σ , the vast majority of variables are 1-frozen w.r.t σ . They did so by stripping down to the $*$ -core, which inspired us to do the same here. One difference between their approach and ours is that the variables of the $*$ -core are 1-frozen by definition, whereas much of the work in this paper is devoted to proving that, for our models, they are in fact $\Theta(n)$ -frozen. We expect that our techniques should be able to prove that the 1-frozen variables established

in [13] are, indeed, $\Theta(n)$ -frozen.

[3] proves the asymptotic (in k) density for the appearance of what they call *rigid* variables in k -COL, k -NAE-SAT and hypergraph 2-colouring (and proves that this is an upper bound for k -SAT). The definition of rigid is somewhat weaker than frozen, but a simple modification extends their proof to show the same for frozen vertices. So [3] provided the asymptotic, in k , location of the freezing threshold for those models. [46] provided the exact location of the threshold for k -COL, when k is sufficiently large.

[4, 3, 48] establish the existence of what they call *cluster-regions* for k -SAT, k -COL, k -NAE-SAT and hypergraph 2-colouring. [3] proves that by the time the density exceeds $(1 + o_k(1))$ times the hypothesized clustering threshold the solution space w.h.p. shatters into an exponential number of $\Theta(n)$ -separated cluster-regions, each containing an exponential number of solutions. While these cluster-regions are not shown to be well-connected, the well-connected property does not seem to be crucial to the difficulties that clusters pose for algorithms. So this was a very big step towards explaining why an algorithmic barrier seems to arise asymptotically (in k) close to the clustering threshold.

[10, 9] provided the first asymptotically tight lower bounds on the satisfiability threshold of k -NAE-SAT and hypergraph 2-colouring, achieving a bound that is roughly equal to the condensation threshold. [24] provides an even stronger bound for hypergraph 2-colouring, extending above the condensation threshold. [23] provides a remarkably strong bound for k -NAE-SAT - the difference between their upper and lower bounds decreases exponentially with k .

2 CSP models

A *boolean constraint* of arity k consists of k *ordered* variables (x_1, \dots, x_k) together with a boolean function $\varphi : \{-1, 1\}^k \rightarrow \{0, 1\}$. This function constrains the set of variables to take values $\sigma = (\sigma_1, \dots, \sigma_k) \in \{-1, 1\}^k$ such that $\varphi(\sigma_1, \dots, \sigma_k) = 1$. We say that the constraint is *satisfied* by a boolean assignment σ if it evaluates to 1 on σ .

A *constraint satisfaction problem (CSP)* is a set of constraints, where the a^{th} constraint is formed by a boolean function φ_a over the variables $(x_{i_{1,a}}, \dots, x_{i_{k,a}})$, with $i_{j,a} \in [n]$. A CSP, H , defines a boolean function $F^{(H)} : \{-1, 1\}^n \rightarrow \{0, 1\}$ given by

$$F^{(H)}(\sigma_1, \dots, \sigma_n) := \prod_a \varphi_a(\sigma_{i_{1,a}}, \dots, \sigma_{i_{k,a}}).$$

Given $\sigma \in \{-1, 1\}^n$, we say that σ is a *satisfying assignment*, or *solution*, of the CSP H if σ satisfies every constraint of H ; i.e. if $F^{(H)}(\sigma) = 1$.

A *CSP model* is a set Φ of boolean functions, together with a probability distribution $p : \Phi \rightarrow [0, 1]$ defined on it (we assume implicitly that the support of p is Φ). Our random CSPs are:

DEFINITION 2.1. *Given a CSP model $\Upsilon = (\Phi, p)$, a random CSP, $C(\Upsilon, n, M)$, is a CSP over the variables $\{x_1, \dots, x_n\}$ consisting of M constraints $\{\varphi_a(x_{i_{1,a}}, \dots, x_{i_{k,a}}) : a = 1, \dots, M\}$ where the boolean constraints $\{\varphi_a : a = 1, \dots, M\}$ are drawn independently from Φ according to the distribution p , and the k -tuples $\{(x_{i_{1,a}}, \dots, x_{i_{k,a}}) : a = 1, \dots, M\}$ are drawn uniformly and independently from the set of k -tuples of $\{x_1, \dots, x_n\}$.*

We consider random CSP-models $\Upsilon = (\Phi, p)$ with the following properties.

DEFINITION 2.2.

Non-trivial: *There is at least one $\varphi \in \Phi$ that is not satisfied by $x_1 = \dots = x_k = 1$ and at least one $\varphi \in \Phi$ that is not satisfied by $x_1 = \dots = x_k = -1$.*

Feasible: *For any $\varphi \in \Phi$, and every assignment to any $k-1$ of the variables, at least one of the two possible assignments to the remaining variable will result in φ being satisfied.*

Symmetric: *For every $\varphi \in \Phi$, and for every assignment x , we have $\varphi(x) = \varphi(-x)$, where $-x$ is the assignment obtained from x by reversing the assignment to each variable.*

Balance-dominated *Consider a random assignment σ where each variable is independently set to be 1 with probability q and -1 with probability $1-q$, and let φ be a random constraint from Φ with distribution p . The probability that σ satisfies φ is maximized at $q = \frac{1}{2}$.*

Those four properties will allow us to apply the planted model. ‘Non-trivial’ is a standard property to require. ‘Feasible’ is also quite natural, although some common models do not satisfy it. The other two properties help us to bound the second moment of the number of solutions, which in turn enables us to use the planted model.

Our final property allows us to analyze frozen variables using the *-core.

DEFINITION 2.3. 1-essential: *Given a boolean constraint φ and an assignment σ that satisfies φ , we say that the variable x is essential for (φ, σ) if changing the value of x results in φ being unsatisfied. We say that a set Φ of constraints is 1-essential if for every $\varphi \in \Phi$, and every σ satisfying φ , at most one variable is essential for (φ, σ) . A CSP-model (Φ, p) is 1-essential if Φ is 1-essential. A CSP is 1-essential if all of its constraints are 1-essential.*

For example: in hypergraph 2-colouring, x is essential iff its value is different from that of every other variable in ϕ ; in k -XOR-SAT, every variable is essential. It is easily confirmed that for $k \geq 3$: k -SAT, hypergraph 2-colouring and k -NAE-SAT are 1-essential, but k -XOR-SAT is not.

3 The planted model

Consider any CSP-model $\Upsilon = (\Phi, p)$. Theorem 1.1 concerns a uniformly random satisfying assignment of $C(\Upsilon, n, M)$; i.e. a pair (F, σ) drawn from:

DEFINITION 3.1. *The uniform model $U(\Upsilon, n, M)$ is a random pair (F, σ) where F is taken from the $C(\Upsilon, n, M)$ model and σ is a uniformly random satisfying solution of F .*

The uniform model is very difficult to analyze directly. So instead we turn to the much more amenable planted model:

DEFINITION 3.2. *The planted model $P(\Upsilon, n, M)$ is a random pair (F, σ) chosen as follows: Take a uniformly random assignment $\sigma \in \{-1, +1\}^n$. Next select a random F drawn from $C(\Upsilon, n, M)$ conditional on σ satisfying F .*

Remark: Note that we can select F by choosing M independent constraints. Each time, we choose a uniformly random k -tuple of k variables, then choose for those variables a constraint $\varphi \in \Phi$ with probability distribution p . If σ does not satisfy the constraint then reject and choose a new one. Equivalently, we can choose the k -tuples non-uniformly where the probability that a particular k -tuple is chosen is proportional to the probability that, upon choosing φ for that set, the constraint will be satisfied by σ . Then we choose $\varphi \in \Phi$ with probability p conditional on σ satisfying φ .

It is not hard to see that the uniform and planted models are not equivalent. In the planted model, a CSP is selected with probability roughly proportional to the number of satisfying assignments. Nevertheless, Achlioptas and Coja-Oghlan [3] proved that, under certain conditions, one can transfer results about the planted model to the uniform model when Υ is k -COL, k -NAE-SAT or hypergraph 2-colouring (also k -SAT, but under stronger conditions). Montanari, Restrepo and Tetali [48] extended this to all Υ in a class of CSP-models, including all models that are non-trivial, feasible, symmetric, and balance-dominated.

For each non-trivial, feasible, symmetric and balance-dominated CSP-model Υ we define (in Appendix 8) a constant $r_p(\Upsilon)$, which is the highest density for which we can use the planted model. The following

key tool essentially follows from Theorem B.3 of [48], except that they do not explicitly mention $r_p(\Upsilon)$, instead giving an implicit lower bound under appropriate conditions. It was first proven in [3] for NAE-SAT, hypergraph 2-COL and a few other models.

LEMMA 3.1. *Consider any non-trivial, feasible, symmetric, and balance-dominated CSP-model Υ . For every $r < r_p(\Upsilon)$, there is a function $g(n) = o(n)$ such that: Let \mathcal{E} be any property of pairs (F, σ) where σ is a satisfying solution of F . If*

$$\Pr(P(\Upsilon, n, M = rn) \text{ has } \mathcal{E}) > 1 - e^{-g(n)},$$

then

$$\Pr(U(\Upsilon, n, M = rn) \text{ has } \mathcal{E}) > 1 - o(1).$$

In Appendix 8, we prove that if Υ is also 1-essential, then for $k \geq 30$, we have $r_p(\Upsilon) > r_f(\Upsilon)$ and so Theorem 1.1 is non-trivial. In fact, $r_p(\Upsilon) = \Theta(\frac{k}{\ln k})r_f(\Upsilon)$. The bound $k \geq 30$ can be lowered, and for some specific models Υ it can be lowered significantly. For example, for k -NAE-SAT and hypergraph 2-colouring, one can probably prove that $k \geq 6$ will do.

4 The *-core

The *-core was introduced in [13] to study frozen variables in random k -SAT.

Fix a satisfying assignment σ , and consider a variable x . Suppose that there are no constraints φ such that x is essential for (φ, σ) . Then, by the definition of essential, we can change x and still have a satisfying assignment. So x is not frozen. This inspires the following:

DEFINITION 4.1. *Consider a CSP F with a satisfying assignment σ . The *-core of (F, σ) is the sub-CSP formed as follows:*

Iteratively remove every variable x such that for every constraint φ : x is not essential for (φ, σ) . When we remove a variable, we also remove all constraints containing that variable.

Note that the order in which variables are deleted will not affect the outcome of the iterative procedure. So the *-core is well-defined, albeit possibly empty.

As described above, it is clear that the first variable removed is not frozen. Expansion properties of a random CSP - in particular the fact that it is locally tree-like - imply that almost every variable removed is not frozen. Furthermore, we will prove that if the model is 1-essential then almost all variables that remain in the *-core are frozen. Having proven those two key

results, Theorem 1.1 follows from an analysis of the *-core process.

Now suppose that our CSP-model is 1-essential. A key observation is that the *-core depends only on the constraints that have essential variables. I.e., if we first remove all constraints with no essential variables from the CSP and then apply the *-core process, the set of variables in the resultant *-core will not change.

DEFINITION 4.2. *Given a 1-essential CSP, F , and a satisfying solution σ , we define the hypergraph $\Gamma(F, \sigma)$ as follows: The vertices are the variables of F and the variables of each constraint of F form a hyperedge, if that constraint has an essential variable. That essential variable is called the essential vertex of the hyperedge.*

Note that we can find the *-core of (F, σ) by repeatedly deleting from $\Gamma(F, \sigma)$ vertices that are not essential in any hyperedges, along with all hyperedges containing the deleted vertices. The resulting hypergraph is called the *-core of $\Gamma(F, \sigma)$.

The precise model for the random hypergraph $\Gamma(F, \sigma)$ varies with Υ (see Appendix 10). However, the size of the *-core as a function of the number of hyperedges is the same for all such models.

We define:

$$\alpha_k := \inf_{x>0} \frac{x}{(1 - e^{-x})^{k-1}}.$$

Also, for $\alpha > \alpha_k$, let $x_k(\alpha)$ be the maximum value of $x \geq 0$ such that $\frac{x}{(1 - e^{-x})^{k-1}} = \alpha$ and set

$$\rho_k(\alpha) = 1 - e^{-x_k(\alpha)}.$$

In the full version of this paper, we prove

LEMMA 4.1. *Consider any 1-essential CSP-model $\Upsilon = (\Phi, p)$ of arity k , and a random CSP, F , drawn from $P(\Upsilon, n, M = rn)$. Suppose $\Gamma(F, \sigma)$ has $\alpha n + o(n)$ hyperedges. For any $g(n) = o(n)$, with probability at least $1 - e^{-g(n)}$:*

- (a) *If $\alpha > \alpha_k$ then the *-core of $\Gamma(F, \sigma)$ has $\rho_k(\alpha)n + o(n)$ vertices.*
- (b) *If $\alpha < \alpha_k$ then the *-core of $\Gamma(F, \sigma)$ has $o(n)$ vertices.*

This allows us to analyze our family of models simultaneously by working directly with the *-core of $\Gamma(F, \sigma)$. We prove that almost all vertices of the *-core are $\Theta(n)$ -frozen variables in F and almost all vertices outside of the *-core are not even 1-frozen in F .

In Appendix 9, we define for any 1-essential CSP-model $\Upsilon = (\Phi, p)$, a constant $\xi(\Upsilon) > 0$ and prove:

LEMMA 4.2. For any $g(n) = o(n)$ and $r > 0$, with probability at least $1 - e^{-g(n)}$, the number of constraints in $P(\Upsilon, n, M = rn)$ that have an essential variable is $\xi(\Upsilon)rn + o(n)$.

Lemmas 4.1, 4.2 yield Theorem 1.1 with:

$$r_f(\Upsilon) = \alpha_k / \xi(\Upsilon); \quad \lambda(\Upsilon, r) = \rho_k(\xi(\Upsilon)r).$$

In Appendix 10, we describe the models that we use to analyze $\Gamma(F, \sigma)$ and the $*$ -core of $\Gamma(F, \sigma)$.

5 Unfrozen variables outside of the $*$ -core

Let x be a vertex of $\Gamma(F, \sigma)$ which is not in the $*$ -core of $\Gamma(F, \sigma)$. We will consider how x can be removed during the peeling process used to find the $*$ -core of $\Gamma(F, \sigma)$. More specifically, we consider a sequence of vertices, culminating in x , which could be removed in sequence by the peeling process.

DEFINITION 5.1. A peeling chain for a vertex $x \in \Gamma(F, \sigma)$ is a sequence of vertices $x_1, \dots, x_\ell = x$ such that each x_i is not essential for any hyperedges in the hypergraph remaining after removing x_1, \dots, x_{i-1} from $\Gamma(F, \sigma)$. The depth of the chain is the maximum distance from one of the vertices to x . The $*$ -depth of x is the minimum depth over all peeling chains for x .

In the full version of this paper, we prove:

LEMMA 5.1. Consider any non-trivial, feasible, symmetric, balance-dominated, and 1-essential CSP-model Υ . Let (F, σ) be drawn from the planted model $P(\Upsilon, n, M = rn)$ where $r \neq r_f(\Upsilon)$. For any $\epsilon > 0$, there exists constant L such that: For all $g(n) = o(n)$, the probability that at least ϵn vertices of $\Gamma(F, \sigma)$ that are not in the $*$ -core of $\Gamma(F, \sigma)$ have $*$ -depth greater than L is less than $e^{-g(n)}$.

This is enough to prove that all but $o(n)$ variables outside the $*$ -core are 1-frozen as follows:

Proof outline of Theorem 1.1(a.ii,b): Consider any $\epsilon > 0$. If (F, σ) is drawn from the planted model then, by Lemma 5.1, $\Gamma(F, \sigma)$ has fewer than ϵn vertices of $*$ -depth greater than L with probability at least $1 - e^{-g(n)}$. So for $r < r_p(\Upsilon)$, Lemma 3.1 yields that the same is true w.h.p. when (F, σ) is drawn from the uniform model.

Consider any vertex x of $*$ -depth at most L . Consider a peeling chain for x of depth at most L and let W be the set of all hyperedges that contain at least one vertex of the peeling chain. If no hyperedges of W form a cycle, then it is easy to see that we can change all of the variables in the peeling chain, one-at-a-time and still

have a satisfying assignment for F . Indeed, this follows from a straightforward induction on L . Therefore, the variable x is not 1-frozen. The case where W contains a cycle is rare enough to be negligible. So for all $\epsilon > 0$ there are fewer than ϵn variables outside of the $*$ -core that are not 1-frozen, as required. \square

This argument also leads to:

Proof outline of Theorem 1.1: This theorem follows as above, by adding the observation that with sufficiently high probability, almost all vertices outside the $*$ -core have a peeling chain of size $O(1)$. We can change the corresponding variable by changing a subset of the entire peeling chain.

For full proofs, see the full version of this paper.

6 Frozen variables in the $*$ -core

Most of the work in this paper is in proving that almost all vertices in the $*$ -core of $\Gamma(F, \sigma)$ are $\Theta(n)$ -frozen. To do so, we first study the structure of sets of variables that can be changed to obtain a new solution. Note that if changing the value of every variable of S yields a solution, then every constraint whose essential variable is in S must contain at least one other variable in S . This leads us to define:

DEFINITION 6.1. A flippable set of the $*$ -core of $\Gamma(F, \sigma)$ is a set of vertices S such that for every $x \in S$ and every $*$ -core hyperedge f in which x is essential, S contains another vertex of f .

For every vertex $x \in S$, since x is in the $*$ -core, there will be at least one such hyperedge f .

Note: if S is a flippable set in $\Gamma(F, \sigma)$, then changing the variables of F corresponding to S will not necessarily yield another solution; this will depend on the actual constraints of F . But it is easily seen that the converse holds:

PROPOSITION 6.1. If σ, σ' are two solutions to a 1-essential CSP, F , then the set of $*$ -core variables on which they differ form a flippable set in $\Gamma(F, \sigma)$.

Proof. Let S be the set of variables in the $*$ -core of (F, σ) on which σ, σ' differ. Suppose that S does not form a flippable set in $\Gamma(F, \sigma)$. Then there is a variable $x \in S$ and a $*$ -core hyperedge e in which x is essential, such that e contains no other members of S . The hyperedge e corresponds to a constraint in F . In that constraint, the solutions σ, σ' agree on all variables but x , which contradicts the fact that x is essential for e .

We prove that for some $\phi'(n) = o(n)$ and constant $\zeta > 0$, with sufficiently high probability, there are no

flippable sets of size $\phi'(n) < a < \zeta n$. This will be enough to prove that at most $o(n)$ vertices lie in flippable sets, which in turn will be enough to show that almost all of the $*$ -core is frozen.

We apply the first moment method. Unfortunately, we cannot apply it directly to the number of flippable sets because the existence of one flippable set S typically leads to the existence of an exponential number of flippable sets formed by adding to S vertices x such that (i) x is essential in exactly one hyperedge, and (ii) that hyperedge contains a non-essential vertex in S . So instead we focus on something that we call *weakly flippable sets*, which do not contain such vertices x . Roughly speaking: every flippable set can be formed from a weakly flippable set by repeatedly adding vertices x in that manner. We prove that with sufficiently high probability:

- (a) There are no weakly flippable sets of size $\phi(n) < a < \zeta n$.
- (b) There are no weakly flippable sets of size at most $\phi(n)$ which extend to a flippable set of size greater than $\phi'(n)$.

This establishes our bound on the sizes of flippable sets. (This is not quite true - we also need to consider *cyclic sets* - but it provides a good intuition.)

Let H_1 denote the vertices that are essential in exactly one hyperedge. Define a *one-path* to be a sequence of vertices x_1, \dots, x_{t+1} such that for each $1 \leq i \leq t$: $x_i \in H_1$ and x_{i+1} is in the hyperedge in which x_i is essential. Note that if x_{t+1} is in a flippable set S , then we can add the entire one-path to S and it will still be flippable. This ends up implying that if we have a proliferation of long one-paths, then we would not be able to prove (b). It turns out that a proliferation of long one-paths would also prevent us from proving (a).

Consider a vertex $x \in H_1$ and the edge f in which x is essential. Intuitively, the expected number of other members of H_1 that are in f is $(k-1)|H_1|$ divided by the size of the $*$ -core. We prove that this ratio is less than 1. This implies that one-paths do not “branch” and so we do not tend to get many long one-paths. So our bound on this ratio plays a key role in establishing both (a) and (b).

This is just an intuition. In fact, *one-paths* are not explicitly mentioned anywhere in the proofs. For all the details, see the full version of this paper.

7 Further Challenges

Of course, one ongoing challenge is to continue to rigorously establish parts of the clustering picture. By now, it is clear that in order to establish satisfiability

thresholds or understand the algorithmic challenges for problems with densities approaching that threshold, we will probably need a strong understanding of clustering.

Another challenge is to try to establish whether the freezing threshold is, indeed, an algorithmic barrier. For several CSP-models, we now know the precise location of that threshold, and we have a very good understanding of how it arises and which variables are frozen. Perhaps we can use that understanding to prove that a simple algorithm works for all densities up to that threshold and/or establish that frozen clusters will indeed necessitate more sophistication.

Another challenge is to determine the freezing threshold for a wider variety of CSP-models. These techniques rely crucially on using the planted model; at this point there is no known way to get to the exact threshold without it. This prevents us from extending our results to k -SAT and many other models as the planted model does not work nearly well enough, mainly because the number of solutions is not sufficiently concentrated. A more important challenge would be to devise a better means to analyze random solutions to CSP's drawn from those models.

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Appendix

8 The transfer theorem

Let us consider a CSP-model $\Upsilon = (\Phi, p)$. Let us recall the properties from Definition 2.2. Given a boolean function $\varphi \in \Phi$, we denote by S_φ the set of satisfying assignments of φ and also we define $I_\varphi := \{-1, 1\}^k \setminus S_\varphi$. Now, let $\varphi(x) = \sum_{Q \subseteq \{-1, 1\}^k} (\varphi_Q \prod_{i \in Q} x_i)$ be its *Fourier expansion*. Such expansion is unique with $\varphi_Q := \sum_{x \in \{-1, 1\}^k} (\varphi(x) \prod_{i \in Q} x_i)$. In particular, it is the case that $\varphi_\emptyset = \frac{|S_\varphi|}{2^k} = \sum_{Q \subseteq \{-1, 1\}^k} \varphi_Q^2$. Moreover, if φ is symmetric, we have that $\varphi_{\{i\}} = 0$ (In fact, $\varphi_Q = 0$ whenever $|Q|$ is odd). Now, we define the polynomial $p_\varphi(\theta)$ as follows,

$$p_\varphi(\theta) := \sum_{Q \subseteq \{-1, 1\}^k} (\varphi_Q / \varphi_\emptyset)^2 \theta^{|Q|}$$

Also, we define the *binary entropy function* $\mathbf{H}(\theta)$ as

$$\mathbf{H}(\theta) := -\frac{1+\theta}{2} \ln(1+\theta) - \frac{1-\theta}{2} \ln(1-\theta)$$

Finally, we define

$$r_p(\Upsilon) := \inf_{\theta \in (0, 1)} \frac{-\mathbf{H}(\theta)}{\sum_{\varphi \in \Phi} p_\varphi \ln(p_\varphi(\theta))}.$$

We will now prove Lemma 3.1, which we restate:

Lemma 3.1 *Consider any non-trivial, feasible, symmetric, and balance-dominated CSP-model Υ . For every $r < r_p(\Upsilon)$, there is a function $g(n) = o(n)$ such that: Let \mathcal{E} be any property of pairs (F, σ) where σ is a satisfying solution of F . If*

$$\Pr(P(\Upsilon, n, M = rn) \text{ has } \mathcal{E}) > 1 - e^{-g(n)},$$

then

$$\Pr(U(\Upsilon, n, M = rn) \text{ has } \mathcal{E}) > 1 - o(1).$$

The proof follows the argument employed in [48] to prove Theorem B.3, which followed the same spirit of similar results in [3].

Proof. In what follows, we will take expectations over a random φ chosen from Φ with distribution p . Thus, for a variable $X(\varphi)$, we have $\mathbf{Exp}(X) = \sum_{\varphi \in \Phi} p(\varphi) X(\varphi)$. Let ξ_φ be the number of clauses with constraint φ in the random CSP H drawn from Υ . Let γ be a fixed constant in $(0, 1/2)$ and let \mathcal{F} be the event ‘For all $\varphi \in \Phi$, $|\xi_\varphi - \alpha p_\varphi n| < n^{1/2+\gamma}$ ’. So, \mathcal{F} holds w.h.p.

We say that a solution σ is *balanced* if the number of variables assigned $+1$ is either $\lceil \frac{n}{2} \rceil$ or $\lfloor \frac{n}{2} \rfloor$. Let Z_b be the number of balanced solutions of H , let Z be the number of solutions of H and let $Z_b(\theta)$ be the number of *pairs* of balanced solutions $x^{(1)}, x^{(2)}$ with discrepancy θ , that is, such that $\frac{1}{n} \sum_{i=1}^n x_i^{(1)} x_i^{(2)} = \theta$. Now,

$$\frac{\mathbf{Exp}[Z_b^2 \mathbf{I}(\mathcal{F})]}{(\mathbf{Exp}[Z_b \mathbf{I}(\mathcal{F})])^2} = \sum_{\theta \in U_n} \frac{\mathbf{Exp}[Z_b(\theta) \mathbf{I}(\mathcal{F})]}{(\mathbf{Exp}[Z_b \mathbf{I}(\mathcal{F})])^2}$$

where $U_n := \{i/n : i = -n, \dots, n\}$. From lemma A.2 in [48], then it is the case that, if $\Phi(\theta) = \mathbf{H}(\theta) + \alpha \mathbf{Exp}[\ln(p_\varphi(\theta))]$

$$\frac{\mathbf{Exp}[Z_b(\theta) \mathbf{I}(\mathcal{F})]}{(\mathbf{Exp}[Z_b \mathbf{I}(\mathcal{F})])^2} \leq C n^{-1/2} \exp(n(\Phi(\theta) + o(1)))$$

where C does not depends on θ (neither the $o(1)$ term).

Now, if $\alpha < r_p(\Upsilon)$, it is the case that

$$(8.1) \quad H(\theta) + \alpha \mathbf{Exp}[\ln(p_\varphi(\theta))] < 0 \text{ for all } \theta \in (0, 1).$$

On the other hand, since Υ is symmetric,

$$\Phi(\theta) = \left(-\frac{1}{2} + \alpha \mathbf{Exp} \left[\sum_{|Q|=2} (\varphi_Q / \varphi_\emptyset)^2 \right] \right) \theta^2 + O(\theta^4).$$

Now, since

$$\lim_{\theta \rightarrow 0} \frac{-H(\theta)}{\mathbf{Exp}[\ln(p_\varphi(\theta))]} = \frac{1/2}{\mathbf{Exp} \left[\sum_{|Q|=2} (\varphi_Q / \varphi_\emptyset)^2 \right]} > \alpha,$$

then it is the case $H(\theta) + \alpha \mathbf{Exp}[\ln(p_\varphi(\theta))] < -c\theta^2$ for some $c > 0$ and θ close enough to 0. Combining this fact with eq. (8.1), we have that for some $c' > 0$,

$$(8.2) \quad H(\theta) + \alpha \mathbf{Exp}[\ln(p_\varphi(\theta))] < -c'\theta^2 \text{ for all } \theta \in (0, 1).$$

Now,

$$\begin{aligned} \frac{\mathbf{Exp}[Z_b^2 \mathbf{I}(\mathcal{F})]}{(\mathbf{Exp}[Z_b \mathbf{I}(\mathcal{F})])^2} &\leq \frac{C}{n^{1/2}} \sum_{\theta \in U_n} \exp(-c'n(\theta^2 + o(1))) \\ &\leq Cn^{1/2} \int_{-\infty}^{\infty} \exp(-c'n(\theta^2 + o(1))) \end{aligned}$$

And the last quantity is bounded by a constant C_0 (not depending on n). This implies, by the Paley-Zygmund inequality, that for every $\epsilon > 0$ and all $n \geq n_0$ it is the case that $\Pr(Z_b > e^{-n\epsilon} \mathbf{Exp}[Z_b]) \geq C_0/2$.

Now, because Υ is balance-dominated, we have that $\mathbf{Exp}[Z] \leq n \mathbf{Exp}[Z_b]$. Therefore, for n large enough, we have that

$$\begin{aligned} \Pr(Z > e^{-n\epsilon} \mathbf{Exp}[Z]) &\geq \Pr(Z_b > ne^{-n\epsilon} \mathbf{Exp}[Z_b]) \\ &\geq \Pr(Z_b \geq e^{-n(\epsilon/2)} \mathbf{Exp}[Z_b]) \\ &\geq C_0/2. \end{aligned}$$

On the other hand, it is easy to see that $\mathbf{Exp}[Z]$ is exponential in n for $\alpha < r_p(\Upsilon)$ (Indeed $\mathbf{Exp}[Z]$ is exponential for $\alpha < r_{\text{sat}}(\Upsilon) := \frac{\ln 2}{\mathbf{Exp}_{\varphi}[\ln(1+|I_{\varphi}|/|S_{\varphi}|)]} = \frac{-H(1)}{\mathbf{Exp}[\ln(p_{\varphi}(1))]}$). Now, let us recall from Appendix C in [48], that the event ' $Z > B^n$ ', where $B > 1$, has a sharp threshold in the clauses to variables ratio. Thus, the event ' $Z > e^{-n\epsilon} \mathbf{Exp}[Z]$ ' has a sharp threshold in the parameter α . Therefore, necessarily, it is the case that $Z > e^{-n\epsilon} \mathbf{Exp}[Z]$ w.h.p.. This implies therefore, that for some function $g(n)$ of order $o(n)$, it is the case that w.h.p.,

$$(8.3) \quad \ln(Z) > \ln(\mathbf{Exp}(Z)) - g(n).$$

After this equation is established now the lemma follows. For instance, from Theorem B.3 in [48].

Now, recall our other property:

1-essential: *Given a boolean constraint φ and an assignment σ that satisfies φ , we say that the variable x is essential for (φ, σ) if changing the value of x results in φ being unsatisfied. We say that a set Φ of constraints is 1-essential if for every $\varphi \in \Phi$, and every σ satisfying φ , at most one variable is essential for (φ, σ) . A CSP-model (Φ, p) is 1-essential if Φ is 1-essential.*

An easy description of a feasible, 1-essential constraint is the following: φ is feasible and 1-essential iff the Hamming distance between any pair of assignments in I_{φ} is greater than 2. This implies in particular that $|I_{\varphi}| \leq \frac{2^k}{\binom{k}{2}+1}$ and $\varphi_{\{i,j\}} = -\frac{1}{2^k} \sum_{x \in I_{\varphi}} x_i x_j$. This allows us to prove a more concrete lower bound on the transfer threshold $r_p(\Upsilon)$ that we will use in the next section to establish that $r_p(\Upsilon)$ is above the freezing threshold for large enough k .

THEOREM 8.1. *Consider any non-trivial, feasible, symmetric, balance-dominated and 1-essential CSP-model Υ . It is the case that*

$$r_p(\Upsilon) \geq \frac{0.25}{\Omega_p(\Upsilon)},$$

where

$$\Omega_p(\Upsilon) := \mathbf{Exp}_{\varphi}[|I_{\varphi}|/|S_{\varphi}|].$$

Proof. Since every constraint $\varphi \in \Phi$ is feasible and 1-essential, we have that

$$\begin{aligned} \sum_{\{i,j\}} \left(\frac{\varphi_{\{i,j\}}}{\varphi_{\emptyset}} \right)^2 &= \sum_{\{i,j\}} \frac{(\sum_{x \in I_{\varphi}} x_i x_j)^2}{|S_{\varphi}|^2} \\ &\leq \binom{k}{2} \left(\frac{|I_{\varphi}|}{|S_{\varphi}|} \right)^2 \end{aligned}$$

Therefore, since

$$\begin{aligned} \sum_{|Q| \geq 4} \varphi_Q^2 \theta^{|Q|} &\leq \sum_{|Q| \geq 4} \varphi_Q^2 \theta^4 \\ &\leq \left(\sum_{Q \subseteq \{-1,1\}^k} \varphi_Q^2 - \varphi_{\emptyset}^2 \right) \theta^4 \\ &= \varphi_{\emptyset} (1 - \varphi_{\emptyset}) \theta^4, \end{aligned}$$

we have that

$$p_{\varphi}(\theta) \leq 1 + \binom{k}{2} \left(\frac{|I_{\varphi}|}{|S_{\varphi}|} \right)^2 \theta^2 + \frac{|I_{\varphi}|}{|S_{\varphi}|} \theta^4$$

And, since $|I_{\varphi}| \leq \frac{2^k}{\binom{k}{2}+1}$, and therefore $\binom{k}{2} \left(\frac{|I_{\varphi}|}{|S_{\varphi}|} \right)^2 \leq \frac{|I_{\varphi}|}{|S_{\varphi}|}$, we get that

$$p_{\varphi}(\theta) \leq 1 + 2 \frac{|I_{\varphi}|}{|S_{\varphi}|} \theta^2$$

Thus,

$$\mathbf{Exp}_{\varphi}[\ln(p_{\varphi}(\theta))] \leq 2\theta^2 \Omega_p(\Upsilon)$$

Now, we finally conclude that

$$\begin{aligned} r_p(\Upsilon) &= \inf_{\theta \in (0,1)} \frac{-H(\theta)}{\mathbf{Exp}_{\varphi}[\ln(p_{\varphi}(\theta))]} \\ &\geq \frac{0.5}{\Omega_p(\Upsilon)} \inf_{\theta \in (0,1)} \frac{-H(\theta)}{\theta^2} = \frac{0.25}{\Omega_p(\Upsilon)}. \end{aligned}$$

We close this section by discussing the CSP-models that satisfy our five conditions: non-trivial, feasible, symmetric, balance-dominated, and 1-essential.

Our properties are rich enough to permit a large class of CSP-models beyond hypergraph 2-coloring and

k -NAE-SAT. For example, we can construct a model in the following way:

Represent the assignments in $\{-1, +1\}^k$ as the k -dimensional hypercube H_k , and so two assignments are adjacent if they differ on exactly one variable. Let L_ϵ denote the vertices $x \in H_k$ with $\sum x_k > \epsilon k$. Consider any subset $I \subseteq L_\epsilon$ containing no two vertices of distance at most two. We use $-I$ to denote the subset formed by switching the sign of every vertex in I , and set $J := I \cup -I$ to be the assignments which violate our constraint φ_J . I.e., $\varphi_J(x) := 1$ iff $x \notin J$.

Now consider any set Φ of constraints of this form in which at least one is non-trivial (i.e. has $(1, 1, \dots, 1) \in J$). Let $\Upsilon = (\Phi, p)$ for any p (such that $\text{supp}(p) = \Phi$). For any k large enough in terms of ϵ , Φ satisfies our five properties. For instance, hypergraph 2-coloring is formed in this way with $I := (1, \dots, 1)$.

Given a constraint φ and some $s \in \{-1, +1\}^k$, we define the constraint φ^s as $\varphi^s(x_1, \dots, x_k) = \varphi(s_1 x_1, \dots, s_k x_k)$. We can allow $\epsilon = 0$ and drop the condition that k must be large if (a) no two vertices of J are within distance 2, and (b) for every $\varphi \in \Phi$ and every $s \in \{-1, +1\}^k$, we have $\varphi^s \in \Phi$ and $p(\varphi^s) = p(\varphi)$. For instance, k -NAE-SAT is formed in this way with $I := (1, \dots, 1)$.

9 Essential hyperedges

Consider any nontrivial, feasible, symmetric 1-essential CSP-model $\Upsilon = (\Phi, p)$. We will draw (F, σ) from the planted model $P(\Upsilon, n, M)$. We begin by taking a random assignment σ for the variables x_1, \dots, x_n and note that $|\Lambda^+|, |\Lambda^-| = \frac{1}{2}n + o(n)$ with probability at least $1 - e^{-g(n)}$, for any $g(n) = o(n)$. So we can assume that this condition holds.

In what follows, we will take expectations over a random φ chosen from Φ with distribution p . Thus, for a variable $X(\varphi)$, we have $\mathbf{Exp}(X) = \sum_{\varphi \in \Phi} p(\varphi) X(\varphi)$.

For every $\varphi \in \Phi$, recall from the previous section that S_φ is the set of assignments in $\{-1, +1\}^k$ that satisfy φ and $I_\varphi = \overline{S_\varphi}$ is the set that do not satisfy φ . We define $S_\varphi^e \subseteq S_\varphi$ to be the set of assignments that satisfy φ and for which φ has an essential variable. Noting that switching the essential variable of an assignment in S_φ^e yields an assignment in I_φ , and using the fact that Υ is feasible, it is easy to see that $|S_\varphi^e| = k|I_\varphi|$.

Since $|\Lambda^+|, |\Lambda^-| = \frac{1}{2}n + o(n)$, it follows that when picking a constraint in the planted model, we choose φ with probability proportional to $p(\varphi)|S_\varphi| + o(1)$. Thus, defining $\Omega_f := \frac{\mathbf{Exp}[|I_\varphi|]}{\mathbf{Exp}[|S_\varphi|]}$, the probability that φ has an essential variable is:

$$\xi(\Upsilon) = k\Omega_f + o(1).$$

So the number of constraints that have an essential variable is distributed as the binomial $BIN(M = rn, \xi(\Upsilon))$. Concentration of the binomial variable implies Lemma 4.2.

Now recall the type of φ , as defined in Section 10. For a constraint $\varphi \in \Phi$, define $I_\varphi(a, b) := \{x \in I_\varphi : x \text{ has } a \text{ 1's and } b - 1\text{'s}\}$ then the clause φ has exactly $(b+1)|I_\varphi(a, b+1)|$ assignments of type $(1; a, b)$ and $(a+1)|I_\varphi(a+1, b)|$ assignments of type $(-1; a, b)$. Therefore, when picking a constraint in the planted model, if we condition on the event that it has an essential variable, then the conditional probability that it has type $\tau = (1; a, b)$ is

$$\gamma_\tau = \frac{(b+1)\mathbf{Exp}[|I_\varphi(a, b+1)|]}{k\mathbf{Exp}[|I_\varphi|]} + o(1)$$

and to be of type $\tau = (-1; a, b)$ is

$$\gamma_\tau = \frac{(a+1)\mathbf{Exp}[|I_\varphi(a+1, b)|]}{k\mathbf{Exp}[|I_\varphi|]} + o(1)$$

Since Υ is symmetric, $\varphi(x) = \varphi(-x)$ for every assignment x . It follows that $|I_\varphi(a, b)| = |I_\varphi(b, a)|$ and therefore $\gamma_{\tau=(1; a, b)} = \gamma_{\tau=(-1; b, a)} + o(1)$. So, noting that we can exchange a, b in the following definition:

$$\gamma^+ := \sum_{\tau=(1; a, b)} \gamma_\tau, \quad \gamma^- := \sum_{\tau=(-1; a, b)} \gamma_\tau,$$

we have $\gamma^+ = \gamma^- = \frac{1}{2} + o(1)$. In other words:

LEMMA 9.1. *When we choose a random clause for the planted model, and condition on it having an essential variable: the probability that the essential variable is in Λ^+ is equal to the probability that it is in Λ^- plus $o(1)$.*

We close this section by showing that $r_f(\Upsilon) < r_p(\Upsilon)$ for sufficiently large k .

PROPOSITION 9.1. *For any nontrivial, symmetric, feasible, balance-dominated, 1 essential CSP model Υ of arity k :*

(a) *For every $k \geq 27$, $r_p(\Upsilon) > r_f(\Upsilon)$.*

(b) *Asymptotically in k , $\frac{r_f(\Upsilon)}{r_p(\Upsilon)} \lesssim \frac{\ln k}{k}$.*

Proof. Notice first that

$$\begin{aligned} \Omega_p &= \mathbf{Exp} \left[\frac{|I_\varphi|}{|S_\varphi|} \right] \leq \frac{\mathbf{Exp}[|I_\varphi|]}{2^k(1 - \frac{1}{\binom{k}{2}+1})} \\ &\leq \frac{\mathbf{Exp}[|I_\varphi|]}{(1 - \frac{1}{\binom{k}{2}+1})\mathbf{Exp}[|S_\varphi|]} = \frac{\Omega_f}{(1 - \frac{1}{\binom{k}{2}+1})}. \end{aligned}$$

Notice also that $\alpha_k \leq \frac{2 \ln(k)}{(1-1/k^2)^{k-1}}$. Therefore, since

$$\frac{2 \ln(k)}{k(1-1/k^2)^{k-1}} \leq (1/4)(1 - \frac{1}{\binom{k}{2} + 1})$$

for $k \geq 27$, then

$$r_f(\Upsilon) \leq \frac{2 \ln(k)}{\Omega_f k(1-1/k^2)^{k-1}} \leq \frac{(1/4)}{\Omega_p} \leq r_p(\Upsilon),$$

by Theorem 8.1. Then, part (a) follows. To prove part (b) we use the previous inequality, so that

$$\frac{r_f(\Upsilon)}{r_p(\Upsilon)} \leq \frac{8 \ln(k)}{k(1 - \frac{1}{\binom{k}{2} + 1})(1-1/k^2)^{k-1}} \sim 8 \ln(k)/k$$

10 Our hypergraph models

Consider any 1-essential CSP, F , and any solution σ .

The vertices of $\Gamma(F, \sigma)$ are partitioned into two sets Λ^+, Λ^- containing those variables which are assigned $+1, -1$ respectively under σ .

DEFINITION 10.1. *For each hyperedge $e \in \Gamma(F, \sigma)$: Let a be the number of non-essential vertices of e in Λ^+ and let b be the number of non-essential vertices of e in Λ^- . The type of e is defined to be:*

- $(1, a, b)$ if the essential vertex vertex of e is in Λ^+ ;
- $(-1, a, b)$, if the essential vertex vertex of e is in Λ^- .

The type of a constraint of (F, σ) with an essential vertex, is the type of the corresponding hyperedge in $\Gamma(F, \sigma)$.

Now consider a nontrivial, feasible, symmetric, balance-dominated, 1-essential CSP-model Υ and choose a random (F, σ) from the planted model $P(\Upsilon, n, M)$. Recalling the Remark following Definition 3.2, we can selected the constraints of F independently. Given the partition Λ^+, Λ^- , and a type τ , we let $w(\tau) = w(\tau, \Lambda^+, \Lambda^-)$ denote the probability that a selected constraint has type τ , conditional on it having an essential vertex. (See Appendix 9 for further discussion.) Note that $w(\tau)$ depends only on $\Upsilon, |\Lambda^+|, |\Lambda^-|$. Note further that, conditional on a hyperedge e having type τ , every choice of the vertices of e which is consistent with τ is equally likely. Thus, when choosing $\Gamma(F, \sigma)$ we can choose the type of a hyperedge first and then its vertices. This leads us to:

Model A:

1. Partition the vertices into Λ^+, Λ^- uniformly at random.
2. For $i = 1$ to M , choose the i th hyperedge e_i as follows:
 - (a) Choose the type (s, a, b) of e_i (where $s \in \{+1, -1\}$), where type τ is chosen with probability $w(\tau)$.
 - (b) Choose the essential vertex for e_i uniformly from the appropriate set, Λ^+ or Λ^- , according to s .
 - (c) Choose a vertices uniformly from Λ^+ and b vertices uniformly from Λ^- . These are the non-essential vertices of e_i .

In some cases, it will be useful to fix the essential vertex of every hyperedge, along with the assignment σ , and then choose our planted hypergraph. In this case, for $s \in \{-1, +1\}$, we use $w^s(\tau) = w(\tau, \Lambda^+, \Lambda^-)$ denote the probability that a selected constraint has type τ , conditional on it having an essential vertex in Λ^s . We can use the following model.

The Essential Model:

1. We are given a partition of the vertices into Λ^+, Λ^- .
2. For $i = 1$ to M , we are given the essential vertex of e_i . We choose the rest of e_i as follows:
 - (a) Choose the type (s, a, b) of e_i , where s is already determined and type τ is chosen with probability $w^s(\tau)$.
 - (b) Choose a vertices uniformly from Λ^+ and b vertices uniformly from Λ^- . These are the non-essential vertices of e_i .

The essential model is useful in analyzing the $*$ -core of $\Gamma(F, \sigma)$.